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Preface

The Vladimir Andrunachievici Institute of Mathematics and Computer Science (VA IMCS) is approaching its 60th anniversary, underscoring its enduring status as a premier research institution in pure and applied mathematics and computer science. Established by the esteemed mathematician Vladimir Andrunachievici, VA IMCS has achieved recognition for its significant scientific contributions and pioneering research schools in algebraic rings, differential equations, quasigroups, functional analysis, and mathematical logic.

Over six decades, the VA IMCS has been a beacon, a guide of knowledge, and a renowned center of excellence. About 80 national and international research projects have been realized; more than 400 PhDs and habilitated doctors theses have been defended.

The institute publishes three prestigious journals: The Bulletin of the Academy of Sciences of the Republic of Moldova. Mathematics, Computer Science Journal of Moldova, and Quasigroups and Related Systems. These three journals make a valuable contribution to the promotion and international recognition of the scientific results obtained by the institute's researchers, ensuring their visibility and facilitating their participation in various international projects.

In celebration of its 60th anniversary, VA IMCS hosted the International Conference, promoting collaboration between academia, research institutions on critical issues in mathematics and computer science and their socio-economic applications. The conference proceedings feature 86 papers spanning Pure Mathematics, Applied Mathematics, and Computer Science, focusing on practical solutions for both national and global challenges.

The anniversary conference, the Proceedings of which we are bringing to your attention, presented a summary of the Institute's achievements and opened new ways for collaboration.

Dr. Inga Ţiţchiev
Director IMCS "Vladimir Andrunachievici", MSU

Plenary Section

Proceedings of the International Conference IMCS-60, MSU
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Statistical Learning for Stochastic PDE models

Igor Cialenco

Abstract

Unlike traditional finite-dimensional stochastic differential equations, statistical models driven by SPDEs are predominantly singular, when the solution is observed on a finite time interval. Hence, conventional inference tools are inadequate and special methods must be developed. After a brief discussion of some classical approaches, we will discuss some classes of parabolic SPDEs for which exploring some specific structures yield intriguing results or unexpected anomalies, e.g., finding parameters bypassing statistical procedures, or introducing nontrivial biases after naïve approximations of estimators. We continue by exploring some cutting-edge methodologies in estimating some of the parameters entering (nonlinear) SPDEs, and conclude with some open problems and possible research directions.

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Choice Principles for Finitely Supported Structures

Gabriel Ciobanu

Abstract

Finitely supported sets are classical sets equipped with actions of the group of permutations of infinitely many basic elements (called atoms) that additionally satisfy a finite support requirement. These sets can be successfully used to study infinite structures that contain "enough" symmetries such that they can be concisely manipulated. In this talk, there are presented the foundations of finitely supported structures. Finitely supported structures are related to permutation models of Zermelo-Fraenkel set theory with atoms (ZFA) introduced in 1930s by Fraenkel, Lindenbaum, and Mostowski to prove the independence of the axiom of choice and the other axioms in ZFA. We use a set theory defined by the axioms of ZFA set theory extended with an additional axiom for finite support. The consistency of choice principles in various models of Zermelo-Fraenkel set theory (ZF) and ZFA (including the symmetric models and the permutation models) was investigated deeply in the last century; the choice axiom was proved to be independent from the axioms of ZF and ZFA. In the new framework of finitely supported structures, we analyze the validity of various choice principles (and related results), as well as the consistency of results regarding cardinality and infinity. We prove the inconsistency of choice principles for finitely supported structures (i.e., their formulations with respect to the finite support requirement are not valid). Joint work with Andrei Alexandru.

Gabriel Ciobanu

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ChatGPT is not enough. Arguments for a mixed approach in AI

Dan Cristea

Abstract

ChatGPT and other generative solutions spoil themselves in glory. Everybody uses them one way or the other, mostly finding answers to questions, suggestions for programming code, incentives to develop written stories or even drawing sketches that could become visual art. However, the logical capabilities of such systems are known to be yet very limited. It is often the case that they hallucinate when put to give solutions to logical puzzles. Moreover, timelines hidden in complex writings, such as belletristics, are yet difficult to decipher by nowadays AI systems.

Still, classical AI has always been occupied to develop reasoning skills in machines, which seems mostly forgotten in the era of generative transformational approaches. In my speech, I will suggest some possibilities to augment these very new models with sophisticated capacities to interpret texts and to make smart deductions based on texts.

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Model-assisted Adaptive Designs for Dose-ranging Clinical Trials

Vladimir Dragalin

Abstract

The overall goal of dose-ranging studies is to establish the existence, nature, and extent of dose effect. Recently, a great effort has been made to investigate and develop designs and methods for efficient learning about dose response and more accurate and faster decision making on dose. The focus is on adaptive model-assisted optimal designs of experiments modeling safety, efficacy, or both efficacy and safety endpoints. The proposed dose allocation rules are based on a combination of D- and c-optimality criteria in optimal design of experiments.

Keywords: adaptive design, design of experiments, response-adaptive allocation, sequential analysis, stopping rule.

1 Overview

When a new molecular entity is being developed as a potential drug, a dose-ranging study is a critical part of the drug development process before going in to confirmatory clinical trials. The dose-response relationship describes the change in effect on patient caused by differing levels of exposure to a drug. The primary purpose of dose-ranging studies is to determine the dose of a drug for further study in large scale trials. Essentially, the design problem is to allocate efficiently, the limited number of participants at hand to the candidate doses, in order to maximize learning about the dose-response relationship. Traditionally, patients have been allocated to a few selected doses in fixed proportion and the decisions about the best doses were made at the end of the study, based on pairwise comparisons to a control (usually, placebo).

In contrast, an adaptive dose ranging trial design allows for opportunities to learn about the dose-response relationship more fully and efficiently. By adaptive design, we refer to a clinical study design that uses accumulating data to decide how to modify aspects of the study as it continues, without undermining the validity and integrity of the trial (see [16]). The goal of adaptive designs is to learn from the accumulating data and to apply what is learned as quickly as possible. A lot of research has been conducted in the field of the statistical methodology for adaptive designs (see reviews in [3, 8, 9, 11]).

The primary goal of a dose-finding study is to establish the dose-response relationship. The optimal experimental design framework provides enough structure to make this goal attainable. It is assumed that the available doses (the design region) and the response variables have been defined and there exists a known structure for the mathematical model describing the dose-response relationship (the model). The objective is on choosing the dose levels in some optimal way to enhance the process of estimating the unknown parameters of the model θ. The experimental designs are represented by a set of design points and a corresponding set of weights representing the allocations to the design points: $\xi = \{(d_i, w_i), i = 1, k\}$. An important element in optimal design is the Fisher information matrix, say $M(\xi, \theta)$, which is an expression of the accuracy of the θ estimate based on observations at k design points of design ξ. A "larger" value of M reflects more information (more precision, lower variability) in the estimate. A natural goal in picking the design ξ is to find the design that "maximizes" the determinant of matrix M, the so-called the D-optimal criterion.

A major challenge in design for nonlinear (in θ) models is that the optimal design ξ^* depends on θ – a conundrum: one is looking for the design ξ with the aim of estimating the unknown θ, and yet one has to know θ to find the best ξ. This conundrum leads to various ways of coping with the dependence on θ. These include the locally-optimal design based on one's best guess at θ, Bayesian design by augmenting the criterion to reflect the uncertainty in a prior knowledge about θ, minimax design by finding the design that is optimal under the worst parameter θ value.

The widely used approach in practice is adaptive design that alternates between forming estimates of θ and choosing a locally-optimal design for that value of the parameter at different stages of the trial. Initial design is chosen and preliminary parameter estimates are obtained. Then, the next doses are selected from the available range of doses that satisfy the efficacy and safety constraints and provide the maximal improvement of the design with respect to the selected criterion of optimality and current parameter estimates. The next available cohort of patients is allocated to these doses. The estimates of unknown parameters are refined given these additional observations. These design-estimation steps are repeated until either the available resources are exhausted or the set of acceptable doses is empty. Such an approach is efficient from the perspective of both time and patient resources.

A critical step in the model-assisted approach is positing a model for the data to be observed in the trial. The model can be flexible and have many parameters. In fitting dose-response models, it is only the data that contain information; the model is just an intermediary that is chosen for its flexibility to follow the data. Therefore, the choice of design is far more important than the choice of the model.

The general framework is that a response Y is observed for a given set of parallel groups of subjects corresponding to doses $d_1, d_3, \ldots, d_K$, with d_1 as placebo or active control, for a total of K groups. We consider a one-way layout for the model specification

$$Y_{ij} = \mu_{d_i} + \epsilon_{ij}, \tag{1}$$

where the mean response at dose d_i can be represented as $\mu_{d_i} = f(d_i, \theta)$ for some dose-response model $f(d, \theta)$ parameterized by a vector of parameters θ, and ϵ_{ij} is the error term for patient j within dose group i. We assume for simplicity that $\epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$ are independent identically normally distributed.

The sigmoid E_{max} model is flexible enough to adequately approximate many different families of parametric monotone dose-response models and has been investigated in many recent publications on dose-ranging studies (see [10, 15] and references therein). The mean response has the form

$$f(d, \theta) = \theta_1 + (\theta_2 - \theta_1)\frac{d^{\theta_4}}{d^{\theta_4} + \theta_3^{\theta_4}}. \tag{2}$$

Here $\theta_1 < \theta_2$ and $\theta_3 > 0$. The parameter θ_3, called ED_{50}, is the dose that corresponds to a mean response halfway between the minimum and maximum. The slope parameter θ_4 controls the steepness of the curve. A steeper curve has a slope factor with higher absolute value (the slope is negative for downhill curves) and a shallower curve has a slope factor with lower absolute value of θ_4. This is a highly flexible nonlinear model that captures the essential features of many dose-response relationships such as an apparent threshold dose below which little, if any, treatment effect is observed, an approximately log-linear dose versus mean treatment effect at higher doses, and sometimes a plateau or ceiling effect at relatively high doses. Note that by changing the values of θ_1 and θ_2 and the sign of θ_4, the same model can be used to describe both monotonically increasing and decreasing dose-response curves.

There are several generalizations of the methodology that address different practical situations: two binary endpoints, one for efficacy one for safety ([12, 14, 17], combination of continuous endpoint and binary endpoint [7], combination of two drugs [1, 13], delay response models [2, 6]. Some of these innovative methods have been used in practice, see a short list here [4, 5, 7].

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Products of Random Matrices: from Limit Theorems to Large Deviations

Ion Grama

Abstract

Consider the sequence of independent and identically distributed random matrices $(g_n)_{n\geq 1}$, and the left random walk $G_n := g_n \dots g_1$, $n \geq 1$ on the general linear group $GL(d,\mathbb{R})$. Under appropriate conditions, we derive Bahadur-Rao-Petrov type large deviation expansions for the norm cocycle $|f, G_n v|$ as well as for the coefficient $\langle f, G_n v\rangle$ of the product G_n, where $v \in \mathbb{R}^d$ and $f \in (\mathbb{R}^d)^*$. Additionally, we establish local limit theorems with large deviations. We will present the methods employed in the proofs and discuss applications.

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Stationary Nash Equilibria for Stochastic Positional Games

Dmitrii Lozovanu, Stefan Pickl

Abstract

The problem of the existence and determining stationary Nash equilibria for stochastic positional games with discounted and average payoffs is considered. We show that, for a stochastic positional game with discounted payoffs, there exists a Nash equilibrium in pure stationary strategies and, for a stochastic positional game with average payoffs, there exists a Nash equilibrium in mixed stationary strategies. Some approaches for determining pure and mixed stationary equilibria in such games are proposed.

Keywords: stochastic positional games, average and discounted payoffs, pure and mixed strategies, Nash equilibrium.

1 Introduction and problem formulation

Stochastic games were introduced by Shapley [1]. He considered two-person zero-sum games for which he proved the existence of the value and the optimal stationary strategies of the players with respect to a discounted payoff criterion. Later this result has been extended for m-player stochastic games with discount payoffs of the players [2, 3]. For average stochastic games, stationary Nash equilibria is known to exist only for some special classes of games. In general, the problem of the existence of Nash equilibria for average stochastic games is an open problem, however, for some classes of games (as example for zero-sum average stochastic games), has been proven the existence of Nash equilibria in non-stationary strategies.

In this contribution, we consider a special class of stochastic games with average and discounted payoffs for which Nash equilibria exist. This class of games has been considered in [4] as a generalization of deterministic positional games; we called this class of games *stochastic positional games.* A stochastic positional game with m players is determined by the following elements: A finite set of states X; a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$, where X_i represents the position set of player $i \in \{1, 2, \ldots, m$; a finite set $A(x)$ of actions in each state $x \in X$; a step reward $r^i_{x,a}$ with respect to each player $i \in \{1, 2, \ldots, m\}$ in each state $x \in X$ and for an arbitrary action $a \in A(x)$ and a transition probability function $p : X \times \prod_{x \in X} A(x) \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary $x \in X$ to an arbitrary $y \in X$ for a fixed action $a \in A(x)$, where $\sum_{y \in X} p^a_{x,y} = 1, \forall x \in X, a \in A(x)$.

The positional game starts at the moment of time $t = 0$ in a given state $x_0 \in X$, where the player $i \in \{1, 2, \ldots, m\}$, who is the owner of the state position x_0 ($x_0 \in X_i$), chooses an action $a_0 \in A(x_0)$ and determines the rewards $r^1_{x_0,a_0}, r^2_{x_0,a_0}, \ldots, r^m_{x_0,a_0}$ for the corresponding players $1, 2, \ldots, m$. After that the game passes to a state $y = x_1 \in X$ according to a probability distribution $\{p^{a_0}_{x_0,y}\}$. At the moment of time $t = 1$, the player $k \in \{1, 2, \ldots, m\}$, who is the owner of the state position x_1 ($x_1 \in X_k$), chooses an action $a_1 \in A(x_1)$ and players $1, 2, \ldots, m$ receive the corresponding rewards $r^1_{x_1,a_1}, r^2_{x_1,a_1}, \ldots, r^m_{x_1,a_1}$. Then the game passes to a state $y = x_2 \in X$ according to a probability distribution $\{p^{a_1}_{x_1,y}\}$ and so on indefinitely. Such a play of the game produces a sequence of states and actions $x_0, a_0, x_1, a_1, \ldots, x_t, a_t, \ldots$ that defines a stream of stage rewards $r^1_{x_t,a_t}, r^2_{x_t,a_t}, \ldots, r^m_{x_t,a_t}$, $t = 0, 1, 2, \ldots$. *The average stochastic positional game* is the game with payoffs of the players

$$\omega^i_{x_0} = \lim_{t \to \infty} \inf \; \mathsf{E} \left(\frac{1}{t} \sum_{\tau=0}^{t-1} r^i_{x_\tau, a_\tau} \right), \quad i = 1, 2, \ldots, n,$$

where E is the expectation operator with respect to the probability measure in the Markov process induced by actions chosen by players in their position sets and given starting state x_0. Each player in this game has the aim to maximize his average reward per transition. *The discounted stochastic positional game* is the game with payoffs of the players

$$\omega^i_{x_0,\gamma} = \mathsf{E}\left(\sum_{\tau=0}^{\infty} \gamma^\tau r^i_{x_\tau,a_\tau}\right), \quad i = 1, 2, \ldots, n.$$

2 The main results

A stationary strategy of player $i \in \{1, 2, \ldots, m\}$ is a mapping s^i that provides a probability distribution over the set of actions $A(x_t)$ for every state $x_t \in X_i$. If these probabilities only take values 0 and 1, then s^i is called a pure strategy; otherwise, s^i is called a mixed strategy. A pure strategy of player i is a mapping $s^i : x \to a \in A(x)\ for\ x \in X$ that determines an action $a \in A(x)$ for each state $x \in X$, i.e., $s^i(x) = a$. The set of pure stationary strategies S^i of player i is determined by the set of solutions of the following system

$$\sum_{a\in A(x)} s^i_{x,a} = 1,\ \forall x \in X_i; s^i_{x,a} \in \{0, 1\},\ \forall x \in X_i,\ \forall a \in A(x),$$

and the set of mixed stationary strategies $\mathbf{S}^i$ of player i is determined by the set of solutions of the following system

$$\sum_{a\in A(x)} s^i_{x,a} = 1,\ \forall x \in X_i;\ s^i_{x,a} \geq 0,\ \forall x \in X_i,\ \forall a \in A(x).$$

Let $s = (s^1, s^2, \ldots, s^m)$ be a profile of stationary strategies (pure or mixed strategies) of the players. Then the probability transition matrix $P^s = (p_{x,y})$ in the Markov process induced by s can be calculated as follows $p^s_{x,y} = \sum_{a\in A(x)} s^i_{x,a} p^a_{x,y},\ x \in X^i,\ i = 1, 2, \ldots, m$. If $Q^s = (q^s_{x,y})$ is the limiting matrix of P^s, then the average payoffs $\omega^1_{x_0}(s), \omega^2_{x_0}, \ldots, \omega^m_{x_0}(s)$ of the corresponding players in a positional game are defined as follows: $\omega^i_{x_0}(s) = \sum_{k=1}^m \sum_{y\in X_k} q^i_{x_0,s^k},\ i = 1, 2, \ldots, m$, where $r^i_{y,s^k} = \sum_{a\in A(y)} s^k_{y,a} r^i_{y,a}$, for $y \in X_k,\ k \in \{1, 2, \ldots, m\}$. The discounted payoffs $\sigma^1_{x_o}(s), \sigma^2_{x_o}, \ldots, \sigma^m_{x_o}(s)$ of the players in a positional game are defined as $\sigma^i_{x_0} = \sum_{y\in X} w_{x_0,y} r^i_{y,s},\ i = 1, 2, \ldots, m$, where $w^s_{x,y}$ represent the elements of the matrix $W^s = (I - \gamma P^s)$.

We proved the following two theorems.

Theorem 1. *The game in normal form* $\langle \{S^i\}_{i=\overline{1,m}}, \{\omega^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ *has a Nash equilibrium which is a Nash equilibrium in mixed stationary strategies for the average stochastic positional game.*

Theorem 2. *The game in normal form* $\langle \{S^i\}_{i=\overline{1,m}}, \{\sigma^i_{x_0}(\mathbf{s})\}_{i=\overline{1,m}} \rangle$ *has a Nash equilibrium which is a Nash equilibrium in pure stationary strategies for the discounted stochastic positional game.*

3 Conclusion

For average and discounted stochastic positional games there exist Nash equilibria in stationary strategies.

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Bol-Moufang Rings and things

J. D. Phillips

Abstract

Theorems. Let L be a left C loop. Then, $N\lambda(L) = N\mu(L)$ is normal in L.

The quotient of a C loop by its nucleus is a Steiner loop.

The quotient of a left C loop by its left nucleus is a left Steiner loop.

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Algebro-Geometric and Diophantine Aspects of Integrability in the Theory of Polynomial Vector Fields

Dana Schlomiuk

Abstract

Planar polynomial vector fields intervene in many applications and they also have theoretical importance. There are several open problems on these systems formulated over one hundred years ago, among them the second part of Hilbert's 16th problem and the problem of Poincaré.

While Hilbert's 16th problem is essentially transcendental involving mainly analytic aspects, the problem of Poincaré is of an algebraic geometric nature.

Its roots are in the theory of integrability of Darboux in terms of algebraic invariant curves.

Except for the work of Poincaré, where he formulated this problem, and work of Painlevé and Autonne, all published at the end of the 19th century, the theory of Darboux remained almost dormant for over a century. During the last decades, however, we witnessed a revival of this theory resulting in an extention of Darboux's theory with introduction of new concepts, formulation of new problems and proofs of new results.

In this talk we highlight some of these developments involving apart from algebro-geometric aspects also Diophantian ones.

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Alternative Clifford-like algebras

Jonathan D. H. Smith

Abstract

The real, complex, and quaternion division algebras may be constructed as Clifford algebras. However, while the octonions and split octonions share many features with Clifford algebras, such as an involutary anti-automorphism, their nonassociativity prevents their realization as Clifford algebras. We now introduce *Kingdon algebras*: alternative Clifford-like algebras over vector spaces which are equipped with a symmetric bilinear form, such that the octonions and split octonions arise as Kingdon algebras. In the process, they acquire 2-graded (superalgebra) structure. A comparable construction yields 3-graded algebras over spaces equipped with an alternating trilinear form.

Keywords: Clifford algebra, Kingdon algebra, alternative algebra, graded algebra, trilinear form.

1 Introduction

The algebra in this paper rests on some geometric definitions.

Definition 1. Consider a field F (not of characteristic 2), and a vector space V over F.

(a) A (*bilinearly*) *formed space* (V, B) is a vector space over F with a symmetric bilinear form $B\colon V \times V \to F$.

(b) A trilinear form $T\colon V \times V \times V \to F$ is said to *alternate* whenever $T(x_{1\pi}, x_{2\pi}, x_{3\pi}) = (\operatorname{sgn} \pi) T(x_1, x_2, x_3)$ for each permutation π of $\{1, 2, 3\}$.

(c) A *trilinearly formed space* (V, T) is a vector space over F with an alternating trilinear form $T\colon V \times V \times V \to F$.

(d) A *doubly formed space* (V, B, T) is a bilinearly formed space with an alternating trilinear form.

The *Clifford algebra* $\mathrm{Cl}(V, B)$ over a formed space (V, B) is the quotient of the free associative algebra or *tensor algebra* TV over V by the ideal

$$J_B = \langle uv + vu - B(u, v) | u, v \in V \rangle \tag{1}$$

of TV [1, p.102]. While $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$ all appear as Clifford algebras, the octonions $\mathbb{O}$ or split octonions $\widetilde{\mathbb{O}}$ do not, as they are not associative. Nevertheless, they have recently [2, 3] been given a Clifford-like construction (still with a $\mathbb{Z}/_2$-grading) over formed spaces (V, B), as shown in §2 below. In [2, Ch. 3], the Clifford-like construction was extended to ungraded algebras over doubly formed spaces, which actually yield $\mathbb{Z}/_3$-graded algebras over trilinearly formed spaces (§3).

2 Kingdon algebras

For elements x, y, z of a binary algebra A, the (*additive*) *associator* is the element $(x, y, z) = (xy)z - x(yz)$ of A. The algebra A is said to be *alternative* if the associator is alternating, in the sense that $(x_{1\pi}, x_{2\pi}, x_{3\pi}) = (\operatorname{sgn} \pi)(x_1, x_2, x_3)$ for all permutations π of $\{1, 2, 3\}$. The octonions $\mathbb{O}$ and split octonions $\widetilde{\mathbb{O}}$ are alternative.

Over a given field F, write $\mathrm{Alt}[V]$ for the free alternative algebra over V [4, §1.2]. The *Kingdon algebra* $K(V, B)$ over a formed space (V, B) is the quotient of the free alternative algebra $\mathrm{Alt}[V]$ over V by the ideal

$$I_B = \langle uv + vu - B(u, v), (uv)w - w(vu) | u, v, w \in V \rangle \tag{2}$$

of $\mathrm{Alt}[V]$ [3, Def'n 3.1]. The comparison between the ideals (2) and (1) is immediate. In particular, if the dimension of (V, B) is less than 3, then $\mathrm{Alt}[V] = TV$ and $I_B = J_B$, so $K(V, B) = \mathrm{Cl}(V, B)$ [3, Prop. 3.9]. Kingdon algebras inherit the $\mathbb{Z}/_2$-grading of $\mathrm{Alt}[V]$ [3, Prop. 3.2].

Theorem 2. *[3, Th. 3.14] Consider a three-dimensional vector space V spanned by $\{i, j, k\}$ over a field F which is not of characteristic 2. Suppose that the set $\{i, j, k\}$ is orthogonal with respect to a symmetric bilinear form B on V.*

(a) *If $B(i,i) = B(j,j) = B(k,k) = -2$, then $K(V,B) \cong \mathbb{O}$ over F.*

(b) *If $B(i,i) = B(j,j) = -2, B(k,k) = 2$, then $K(V,B) \cong \widetilde{\mathbb{O}}$ over F.*

3 Ternary forms and algebras

The *BT-Kingdon algebra* $K(V,B,T)$ over a doubly formed space (V,B,T) is the quotient of $\mathrm{Alt}[V]$ over V by the ideal

Table 1. Multiplication table of a T-Kingdon algebra $K(V,T)$ with basis $\{1, i, j, k, ij, jk, ki, \omega\}$ over a trilinearly formed space (V,T) with basis $\{i, j, k\}$ (compare [2, Table 3.2]). The first header row gives the $\mathbb{Z}/_3$ degrees of the column label elements that appear in the second header row below them. Here 1 is the unit, $\omega = (ij)k$, $T = T(i,j,k)$.

	1	1	1	2	2	2	0
	i	j	k	ij	jk	ki	ω
i	0	ij	$-ki$	0	$T-\omega$	0	Ti
j	$-ij$	0	jk	0	0	$T-\omega$	Tj
k	ki	$-jk$	0	$T-\omega$	0	0	Tk
ij	0	0	ω	0	$-Tj$	Ti	0
jk	ω	0	0	Tj	0	$-Tk$	0
ki	0	ω	0	$-Ti$	Tk	0	0
ω	0	0	0	Tij	Tjk	Tki	$T\omega$

$$I_{B,T} = \langle uv + vu - B(u, v), (uv)w - w(vu) - T(u, v, w) | u, v, w \in V \rangle$$

of Alt$[V]$ [2, Def'n 3.3.1]. The ideal $I_{B,T}$ is inhomogeneous with respect to the $\mathbb{Z}/_2$-grading of Alt$[V]$ whenever the alternating trilinear form T is nonzero, so the algebra $K(V, B, T)$ does not carry a $\mathbb{Z}/_2$-grading in that case.

On the other hand, if the bilinear form vanishes, then the ideal $I_{B,T}$ is homogeneous with respect to the $\mathbb{Z}/_3$-grading of Alt$[V]$. In this case, the BT-Kingdon algebra $K(V, B, T)$ is defined to be the *T-Kingdon algebra* $K(V, T)$ of the trilinearly formed space (V, T). Such algebras do inherit the $\mathbb{Z}/_3$-grading, as exhibited in the example of Table 1.

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Subalgebras of Polynomial Algebras (and their Problems)

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Erik Lefler, Erik Kennerland

Abstract

We consider different constructions of subalgebras in polynomial algebra, discuss their invariant properties and some open problems.

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Computational Analysis of Viral Sequencing Data

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Section 1

Pure Mathematics

Section 1.1

Algebra & Topology

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Finiteness for Sets with Atoms

Andrei Alexandru, Gabriel Ciobanu

Abstract

We present various definitions for finite sets in the framework of finitely supported sets. We compare these definitions and emphasize their independence.

Keywords: finitely supported sets; several independent definitions of finite sets with atoms.

1 Finitely Supported Sets

Finitely supported structures [1] are related to the permutation models of Zermelo-Fraenkel set theory with atoms (ZFA). More recently, finitely supported sets have been developed in Zermelo-Fraenkel (ZF) set theory by equipping classical ZF sets with actions of a group of finite permutations of some basic elements called atoms that can only be compared for equality. These atomic sets were used to study the renaming of the bound variables by fresh names in the theory of programming, as well as to describe automata, languages and Turing machines operating over infinite alphabets. The theory of finitely supported algebraic structures [1] allows a discrete representation of possibly infinite structures containing enough symmetries to be concisely handled. More exactly, this theory allows us to treat as equivalent the objects of a structure that have a certain degree of similarity; thus, we focus only on those objects that are 'really different' by using the notions of 'finite support'.

Let us consider a set A of infinitely many atoms. We denote by P_A the set of all bijections of A permuting only finitely many atoms. A

P_A-*set* is a pair $(X, \cdot)$, where X is a ZF set and $\cdot$ is a group action of P_A on X. If $(X, \cdot)$ is a P_A-set, we say that a finite subset S of atoms supports x (namely, x is *finitely supported*) if for each $\pi \in P_A$ fixing pointwise any element of S, we have $\pi \cdot x = x$. If $(X, \cdot)$ is a P_A-set, we say that X is an *invariant set* whenever each $x \in X$ is finitely supported. If there is a finite set supporting x, then there exists a least finite set $supp(x)$ supporting x which is called *the support of* x. An empty supported element is *equivariant*.

Let $(X, \cdot)$ and $(Y, \triangleright)$ be P_A-sets. According to [1], the set A of atoms is an invariant set with the P_A-action $\diamond$ defined by $\pi \diamond a := \pi(a)$ for all $\pi \in P_A$ and $a \in A$. Non-atomic sets are trivially invariant, i.e. they are equipped with the action $\diamond$ defined as $(\pi, x) \mapsto x$. The Cartesian product $X \times Y$ is a P_A-set with the P_A-action $\otimes$ defined by $\pi \otimes (x, y) = (\pi \cdot x, \pi \triangleright y)$ for all $\pi \in P_A$ and all $x \in X$, $y \in Y$. The disjoint union $X + Y = \{(0, x) \,|\, x \in X\} \cup \{(1, y) \,|\, y \in Y\}$ is a P_A-set with the P_A-action $\star$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \triangleright y)$ if $z = (1, y)$. For $(X, \cdot)$ and $(Y, \triangleright)$ invariant sets, $(X \times Y, \otimes)$ and $(X + Y, \star)$ are also invariant sets. The powerset $\mathcal{P}(X)$ is a P_A-set with the P_A-action $\star$ defined by $\pi \star Z := \{\pi \cdot z \,|\, z \in Z\}$ for all $\pi \in P_A$ and $Z \subseteq X$. We denote by $\mathcal{P}_{fs}(X)$ the set formed from those subsets of X that are finitely supported as elements in $\mathcal{P}(X)$ with respect to the action $\star$. A subset Z of an invariant set $(X, \cdot)$ is called *finitely supported* if and only if $Z \in \mathcal{P}_{fs}(X)$. A subset Z of an invariant set $(X, \cdot)$ is *uniformly supported* if there is a finite set of atoms S such that $supp(x) \subseteq S$ for all $x \in X$. A function $f : Z \to T$ is *finitely supported* if $f \in \mathcal{P}_{fs}(X \times Y)$. The set of all finitely supported functions from Z to T is denoted by T^Z_{fs}. We have that Y^X is a P_A-set with the P_A-action $\widetilde{\star}$ defined by $(\pi \widetilde{\star} f)(x) = \pi \triangleright (f(\pi^{-1} \cdot x))$ for all $\pi \in P_A$, $f \in Y^X$ and $x \in X$. A function $f : Z \to T$ is supported by a finite set $S \subset A$ if and only if for all $x \in Z$ and all $\pi \in P_A$ fixing S pointwise we have $\pi \cdot x \in Z$, $\pi \triangleright f(x) \in T$ and $f(\pi \cdot x) = \pi \triangleright f(x)$. Two finitely supported sets X and Y are called *equipollent* if there exists a finitely supported bijection between them. The equipollence relation is an equivariant equivalence relation on the family of all finitely supported sets [2].

The *cardinality* of X is the equivalence class of all finitely supported sets that are equipollent to X, and is denoted by $|X|$. Two finitely supported sets X and Y have the same cardinality (i.e., $|X| = |Y|$) if and only if there exists a finitely supported bijection $f : X \to Y$.

2 Several Definitions of Finiteness

We present several non-equivalent definitions of finite in the framework of finitely supported sets, generalizing the classical approach in the ZF set theory [3] and provide various relationship results between them.

Definition 1. *Let X be a finitely supported set:*

1. *X is called* 1-finite *if $|X \times X| \neq |X|$.*
2. *X is called* 2-finite *if $|X + X| \neq |X|$.*
3. *X is called* 3-finite *if every finitely supported injective function $f : X \to X$ is also surjective (or, equivalently, there is no finitely supported injective function from $\mathbb{N}$ to X).*
4. *X is called* 4-finite *if every finitely supported chain of X is finite.*
5. *X is called* 5-finite *if every uniformly supported subset of X is finite.*
6. *X is called* 6-finite *if for every finitely supported countable chain of finitely supported sets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_i \subseteq \ldots \subseteq X$ is stationary.*
7. *X is called* 7-finite *if every finitely supported family of finitely supported subsets of X, totally ordered with respect to the inclusion relation, has a maximal element.*
8. *X is called* 8-finite *if any finitely supported subset of X is either finite or it has a finite complement.*
9. *X is called* 9-finite *if X is can be expressed as $\{x_1, \ldots, x_n\}$.*

We establish several implications between the different forms of finiteness in the framework of finitely supported sets.

Theorem 2. *Let X be a finitely supported set.*

- *X is 9-finite $\Rightarrow$ X is 7-finite $\Rightarrow$ X is 4-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*

- *X is 9-finite $\Rightarrow$ X is 5-finite $\Rightarrow$ X is 4-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*

- *X is 9-finite $\Rightarrow$ X is 8-finite $\Rightarrow$ X is 6-finite $\Leftrightarrow$ $\mathcal{P}_{fs}(X)$ is 3-finite $\Rightarrow$ X is 3-finite $\Rightarrow$ X is 2-finite $\Rightarrow$ X is 1-finite;*

- *X is 9-finite $\Rightarrow$ X is 7-finite $\Rightarrow$ X is 6-finite;*

- *X is 5 -finite $\Rightarrow$ $\mathcal{P}_{fin}(X)$ is 5-finite, where $\mathcal{P}_{fin}(X)$ is the family of all finite subsets of X;*

- *X is 5 -finite $\Rightarrow$ $X_{fs}^{A^n}$, where A^n (the n-times Cartesian product of A) and X_{fs}^{nA} (where nA is the n-times disjoint union of A) are 5-finite;*

- *X is 5 -finite $\Rightarrow$ $X_{fs}^{\mathcal{P}_n(A)}$ is 5-finite, where $\mathcal{P}_n(A)$ is the set of all n-sized subsets of A.*

3 Independence of Several Definitions of Finite

We present examples of some finitely supported sets satisfying certain definitions of finite while they do not satisfy others, emphasizing in this way the independence of various definitions for finite sets. In this table, we assume that X is 5 -finite, Y has the property that $\mathcal{P}_{fs}(Y)$ is 5-finite (with $|X|, |Y| \geq 2$), and consider Z to be one of the sets kA or A^k for some $k \in \mathbb{N}$, $k > 0$.

Set	1 f	2 f	3 f	4 f	5 f	6 f	7 f	8 f	9 f
A;	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
nA, A^n, $n>1$	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No
$\mathcal{P}_{fs}(Z)$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_n(A))$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(Z)_{fs}^{\mathcal{P}_n(A)}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fs}(A)_{fs}^{Z}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
X_{fs}^{Z}	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$\mathcal{P}_{fin}(X)$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$(\mathcal{P}_{fs}(Z)_{fs}^{Y}$	Yes	Yes	Yes	Yes	Yes	No	No	No	No
$A\cup\mathbb{N}$	Yes	Yes	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(A\cup\mathbb{N})$	Yes	No	No	No	No	No	No	No	No
$A\times\mathbb{N}$	Yes	No	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_{fs}(A))$	?	No	No	No	No	No	No	No	No
$\mathcal{P}_{fs}(\mathcal{P}_{fs}(\mathcal{P}_{fs}(A)))$	No	No	No	No	No	No	No	No	No

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Factor-Group of a Topological Group and Factor-Ring of a Topological Ring and Completeness Issues

V. I. Arnautov, G. N. Ermakova

Abstract

The connection between the completeness of topological groups and topological rings and the completeness of their factor groups and factor rings, respectively, are discussed in this article.

Keywords: Separable topological group, bounded set in a topological ring, locally bounded topological ring, Cauchy sequence over a directed set, limit of a sequence over a directed set, complete topological group.

1 Introduction

Preserving the completeness of topological groups and the completeness of topological rings under various constructions is one of the areas of research in topological algebra (see, for example, [1]).

The question of preserving the completeness of topological groups and topological rings when taking factor groups and factor-rings, respectively, is considered in this work.

2 Basic definitions

This section outlines some well-known concepts that are necessary to present the main results.

Notation 2.1. If (X, τ) is a topological space and $A \subseteq X$, then we denote the restriction of the topology τ to the set SAS by $\tau|_A$.

Definition 2.1. As usual, a partially ordered set $(\Gamma, <)$ is called *a directed set* if for any two elements γ_1 and γ_2 from Γ there exists an element γ_3 such that $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_3$.

Definition 2.2. If $(\Gamma, <)$ is some directed set, then, as usual, a sequence $\{g_\gamma | \gamma \in \Gamma\}$ of elements of a topological Abelian group $(G(+), \tau)$ is called *a Cauchy sequence over a directed set* $(\Gamma, <)$ in a topological group $(G(+), \tau)$ if for any neighborhood V of zero of a topological group $(G(+), \tau)$ there exists an element $\gamma_0 \in \Gamma$ such that $\gamma - \gamma_0 \in V$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$.

Definition 2.3. If $(\Gamma, <)$ is a directed set, then an element g of a topological Abelian group $(G(+), \tau)$ is called the *limit of the Cauchy sequence* $\{g_\gamma | \gamma \in \Gamma\}$ *over this directed set* if for any neighborhood V of zero in the topological group $(G(+), \tau)$ there is an element γ_0 such that $g - g_\gamma \in V$ for any $\gamma \in \Gamma$ such that $\gamma \geq \gamma_0$.

In this case, we will write that $g = \lim_{\gamma \in \Gamma} |\{g_\gamma | \gamma \in \Gamma\}$.

Definition 2.4. A topological group $(G(+), \tau)$ is called a *complete group* if for any directed set $(\Gamma, <)$, any Cauchy sequence $\{g_\gamma | \gamma \in \Gamma\}$ over this directed set has a limit.

Definition 2.5. A topological group $(G(+), \tau)$ is called *separable* if its topology is Hausdorff.

Definition 2.6. A subset S of a topological ring (R, τ) is called *bounded* if for any neighborhood V of zero of the topological ring (R, τ) there is a neighborhood U of zero of the topological ring (R, τ) such that $S \cdot U \subseteq V$ and $U \cdot S \subseteq V$.

Definition 2.7. A topological ring is called *locally bounded* if it has a neighborhood of zero that is a bounded set.

3 Main results

This section presents results that address this issue; some of them were previously published in other articles (see [1] and [2]) but are included here for completeness.

Theorem 3.1. *For any separable topological Abelian group* $(G(+), \tau)$, *there exist a complete separable topological Abelian group* $(\widetilde{G}(+), \widetilde{\tau})$ *and a closed subgroup* N *in the topological group* $(\widetilde{G}(+), \widetilde{\tau})$, *such that the factor group* $(\widetilde{G}(+), \widetilde{\tau})/N$ *and the topological group* $(G(+), \tau)$ *are isomorphic.*

Remark 3.1. Unfortunately, there is an error in the proof of the analogue of Theorem 3.1 for topological rings in the monograph [1]. The main difficulty is to construct the corresponding complete topological ring. This error for locally bounded topological rings is corrected by the following theorem.

Theorem 3.2. For any separable locally bounded topological ring (R, τ), there exist a complete separable topological ring $(\widetilde{R}, \widetilde{\tau})$ and a closed ideal N in the topological ring $(\widetilde{R}, \widetilde{\tau})$ such that the factor ring $(\widetilde{R}, \widetilde{\tau})/N$ and the ring (R, τ) are isomorphic.

Theorem 3.3. *Let* N *be a closed subgroup of a commutative group* $G(+)$ *and let* τ *be a separable group topology on a group* $G(+)$. *If the topological group* $(N, \tau|_N)$ *and the factor group* $(G(+), \tau)/N$ *are complete groups, then the topological group* $(G(+), \tau)$ *is also a complete group.*

Theorem 3.4. *If a separable, complete topological commutative group* (G, τ) *satisfies the first axiom of countability (i.e., it has a countable basis of the filter of neighborhoods of zero), then the factor group* $(G, \tau)/N$ *is a complete group for any closed subgroup* N *of the topological group* (G, τ).

Remark 3.2. Since the completeness of a topological ring is determined by the completeness of its additive group, then for topological rings analogues of Theorems 3.3, 3.4, and 3.5 are true.

Remark 3.3. Since from Theorems 2.1 and 2.2 it follows that the factor group of a complete topological group and the factor ring of a complete topological ring may not be complete. Therefore, it is interesting to obtain conditions when the completeness of the factor group and the factor ring is preserved. The next theorem addresses this issue.

Theorem 3.5. *If M is a subgroup of an Abelian group $G(+)$ and τ is such a group topology that (G, τ) is a complete separable topological group and M is a discrete subgroup of the topological group (G, τ), then the factor group $(G, \tau)/M$ is a complete group.*

Corollary. *If I is an ideal of a ring R and τ is a ring topology on the ring R such that it is a complete separable topological ring and $(I, \tau|_I)$ is a discrete ideal of the topological ring (R, τ), then the factor ring $(R, \tau)/I$ is a complete topological ring.*

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On Geodesic Points of 6-Dimensional Hermitian Submanifolds of Cayley Algebra

Mihail Banaru, Galina Banaru

Abstract

Six-dimensional Hermitian submanifolds of Cayley algebra defined by means of three-fold vector cross products are considered. It is proved that the Weyl tensor of a 6-dimensional Hermitian submanifold of Cayley algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.

Keywords: geodesic point, Hermitian manifold.

1 Introduction

The geometry of 6-dimensional almost Hermitian submanifolds of Cayley algebra has been developing since the 60s of the last century. Among the many famous mathematicians who worked in this field, we can especially highlight the outstanding American specialist Alfred Gray and the Russian geometer Vadim Feodorovich Kirichenko. V.F. Kirichenko, among other results, obtained a complete classification of 6-dimensional Kählerian submanifolds of the octave algebra [1]. Note that the survey [2] contains a significant part of the achievements in the field of geometry of 6-dimensional almost Hermitian submanifolds of octave algebra (naturally, without results obtained over the past 10 years).

In the present communication, we consider geodesic points of 6-dimensional Hermitian submanifolds of Cayley algebra, i.e. the points at which the configuration tensor vanishes. We prove that the Weyl tensor of conformal curvature of a 6-dimensional Hermitian submanifold of the octave algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.

2 Preliminaries

Let us consider an almost Hermitian manifold, i.e. an even-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold $\langle JX, JY \rangle = \langle X, Y \rangle, \quad X, Y \in \aleph(M^{2n})$, where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}.

We recall that the fundamental form of an almost Hermitian manifold is determined by the relation $F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n})$.

As it is known, an almost Hermitian manifold is Hermitian, if its almost complex structure is integrable [3].

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. It is known [4], two non-isomorphic three-fold vector cross products are defined on it by means of the relations:

$$P_1(X, Y, Z) = -X(\bar{Y}Z) + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$

$$P_2(X, Y, Z) = -(X\bar{Y})Z + \langle X, Y \rangle Z + \langle Y, Z \rangle X - \langle Z, X \rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbf{O}$ and $X \to \bar{X}$ is the conjugation operator. If $M^6 \subset \mathbf{O}$ is a 6-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_t, g = \langle \cdot, \cdot \rangle\}$ is determined by the relation $J_t(X) = P_t(X, e_1, e_2)$, $t = 1, 2$, where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at the point p, $X \in T_p(M^6)$ [1]. We recall that the point $p \in M^6$ is called general [4], [5] if $e_0 \notin T_p(M^6)$, where e_0 is the unit of Cayley algebra.

3 The main result

In the above mentioned paper [1], V.F. Kirichenko obtained the Cartan structural equations of a Hermitian structure on a 6-dimensional submanifold of the octave algebra:

$$d\omega^a = \omega^a_b \wedge \omega^b + \frac{1}{\sqrt{2}} \varepsilon^{abh} D_{hc} \omega^c \wedge \omega_b;$$

$$d\omega_a = -\omega_a^b \wedge \omega_b + \frac{1}{\sqrt{2}} \varepsilon_{abh} D^{hc} \omega_c \wedge \omega^b; \qquad (1)$$

$$d\omega_b^a = \omega_c^a \wedge \omega_b^c - \left(\frac{1}{2} \delta_{bg}^{ah} D_{hd} D^{gc} + \sum_\phi T_{\hat{a}\hat{c}}^\phi T_{bd}^\phi \right) \omega_c \wedge \omega^d.$$

Here $\varepsilon_{abc} = \varepsilon_{abc}^{123}$, $\varepsilon^{abc} = \varepsilon_{123}^{abc}$ are the components of the 3-order Kronecher tensor; $D^{hc} = D_{\hat{h}\hat{c}}$, $D_{cj} = \mp T_{cj}^8 + iT_{cj}^7$, $D_{\hat{c}j} = \mp T_{\hat{c}j}^8 - iT_{\hat{c}j}^7$, where $\left\{ T_{kj}^\phi \right\}$ are the components of the configuration tensor (in the Gray–Kirichenko notation); $\phi = 7,\ 8$; $a,\ b,\ c,\ d,\ g,\ h = 1,\ 2,\ 3$; $\hat{a} = a + 3$; $k,\ j = 1,\ 2,\ 3,\ 4,\ 5,\ 6$.

A point $p \in M^6$ is called geodesic if the configuration tensor at this point vanishes.

Let us consider the Weyl tensor (also known as the tensor of conformal curvature) of a 6-dimensional Hermitian submanifold of Cayley algebra. This tensor is defined as follows [3]:

$$W_{ijkl} = R_{ijkl} + \frac{1}{n-2} \left(ric_{ik}\, g_{jl} + ric_{jl}\, g_{ik} - ric_{il}\, g_{jk} - ric_{jk}\, g_{il} \right) +$$

$$+ \frac{K}{(n-1)(n-2)} \left(g_{jk}\, g_{il} - g_{jl}\, g_{ik} \right).$$

Here R is the tensor of Riemannian curvature, ric is the Ricci tensor, K is the scalar curvature and n is the dimension of the manifold.

From Cartan structural equations (1) we can deduce all components of the tensor of Riemannian curvature that determine this tensor (taking into account its symmetry properties):

$$R_{abcd} = 0\,; R_{\hat{a}bcd} = 0\,; R_{\hat{a}\hat{b}cd} = 0\,; R_{\hat{a}b\hat{c}d} = -\sum_\phi T_{\hat{a}\hat{c}}^\phi\, T_{bd}^\phi.$$

Similarly, we obtain the components of the Ricci tensor of 6-dimensional Hermitian submanifold of the octave algebra

$$ric_{ab} = 0; \quad ric_{\hat{a}b} = -\sum_\phi T_{\hat{a}\hat{c}}^\phi\, T_{cb}^\phi.$$

So, we can state our result

Theorem 1. *The Weyl tensor of a 6-dimensional Hermitian submanifold of Cayley algebra vanishes at a geodesic point if and only if the scalar curvature at this point also vanishes.*

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Some Hash Functions Based on Quasigroups

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Abstract

In this paper, we consider the construction of hash functions based on finite quasigroups.

Keywords: quasigroup, Latin square, encryption, one-way function, cryptology, block cipher, symmetric-key encryption.

1 Introduction

Hashing plays a key role in modern information technologies, providing efficient storage, retrieval and integrity checking of data. Traditional hashing methods such as MD5, SHA-1 and their variants, although widely used, face limitations in the case of large data volumes and increased security requirements.

In the recent years, there has been an increased interest in alternative hashing methods based on abstract algebraic structures. Quasigroups and groupoids are important representatives of these structures with unique properties for hashing applications. The high degree of uniqueness of hash values and computational efficiency of their use should be especially noted.

Definitions and basic properties of quasigroups and loops are available in [1, 2, 3].

Definition 1. *A function H() that maps an arbitrary length message M to a fixed length hash value H(M) is a OneWay Hash Function (OWHF), if it satisfies the following properties:*

1.The description of H() is publicly known and should not require any secret information for its operation.

2.Given M, it is easy to compute H(M).

3. Given H(M) in the rang of H(), it is hard to find a message M for given H(M), and given M and H(M), it is hard to find a message $M_0(\neq M)$ *such that* $H(M_0) = H(M)$. [4, 5]

Definition 2. *A OneWay Hash Fucntion H() is called Collision Free Hash Function (CFHF), if it is hard to find two distinct messages M and* M_0 *that hash to the same result* $(H(M) = H(M_0))$. [4, 5]

2 The considered hash function

We give a construction of hashing functions based on quasigroups.

Definition 3. *Let* $H_Q() : Q \to Q$ *be a projection defined as follows:*

$$H_Q(q_1 q_2 ... q_n) = ((...(a \star q_1) \star q_2 \star ...) \star q_n$$

Then $H_Q()$ *is said to be a hash function over the quasigroup* $(Q; \star)$. *The element a is a fixed element from Q.*

Example 1. *Let a quasigroup of order 10 be randomly determined. Suppose that this quasigroup has the following multiplication table:*

⋆	0	1	2	3	4	5	6	7	8	9
0	0	7	9	1	3	4	5	7	2	6
1	7	9	5	2	4	6	3	0	1	8
2	4	1	2	5	6	7	8	9	3	0
3	6	4	0	8	7	5	2	1	9	3
4	3	0	6	9	2	8	7	4	5	1
5	8	2	3	6	9	1	4	7	0	5
6	9	6	4	3	1	2	0	5	8	7
7	1	5	8	4	0	3	6	2	7	9
8	2	8	7	0	5	9	1	3	6	4
9	5	3	1	7	8	0	9	6	4	2

The value of the hash function for the message $M = 0013145123$ would be as follows

$H_2 = ((((((((((2 \star 0) \star 0) \star 1) \star 3) \star 1) \star 4) \star 5) \star 1) \star 2) \star 3) = 3.$

In the first step, we generate a list of quasigroups of order $n, n \in N$. This step is important to ensure the diversity of possible transformations that will be used in the hashing.

Random quasigroups are selected from the generated list. These selected quasigroups will be used in the hashing process to increase persistence and variability of hash values.

For each of the selected quasigroups, the hash value of the original message is calculated. This step allows to obtain intermediate values which will be later combined to form the final hash.

The final step uses a chain hashing method where intermediate hash values are concatenated to produce a final hash value of a given length. This method ensures that the final hash is unique and collision-resistant.

3 Conclusion

In this paper we propose and discuss an approach to constructing a hash function based on quasigroups and the chain hashing method. By randomly selecting quasigroups from a large number of possible variants and using the chain method of combining intermediate hashes, it was possible to create a convenient mechanism for obtaining hash values. In the future, it is planned to further study the optimization of the proposed algorithm and its adaptation for use in systems with higher performance requirements.

Thus, the proposed quasigroup-based hashing method is a promising direction for the development of new cryptographic algorithms and can find wide application in the field of data protection and authentication.

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Basic Properties of the Metric of Tangents

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Abstract

This paper is dedicated to defining the metric of tangents and identifying the class of figures in the plane to which it can be appropriately applied. The concept of a generalized tangent is introduced, generalizing the properties of the support line and the tangent to a smooth curve. The properties of the metric and the class of figures are determined with a view to their practical application and generalization.

Keywords: the metric of tangents, canonical partition, figure with a convex-concave boundary, the generalized tangent

1 Introduction

Currently, there is no metric that would ensure the convergence of shape perimeters, except in trivial cases where one of the terms is the perimeter difference module. We propose such a metric that can be used in practice in image processing.

2 Class of Figures with Convex-Concave Boundary

Let's define the class of figures we will investigate. Let $S^{(1)}$ be the class of planar simply connected regions bounded by closed rectifiable Jordan curves. Consider a figure $\Phi \in S^{(1)}$ and define an orientation ω on its boundary $\Gamma(\Phi)$ such that when moving along the curve $\Gamma(\Phi)$ according to the orientation ω, the figure Φ remains on the right. Denote by $\smile AB$ the arc of the curve $\Gamma(\Phi)$ that connects points A and B lying on $\Gamma(\Phi)$ in the direction of the orientation ω.

Suppose there exists a finite set of points $M_0, M_1, \ldots, M_{n-1}$ that follow in the specified order in the direction of the orientation ω and satisfy the following conditions (here, for convenience, we assume that $M_n = M_0$ and $M_{n+1} = M_1$):

1. The arc $\smile M_iM_{i+1}$ is different from the segment M_iM_{i+1} for $i = \overline{0, n-1}$;

2. The figure F_i bounded by the arc $\smile M_iM_{i+1}$ and the segment M_iM_{i+1} is convex for $i = \overline{0, n-1}$;

3. The figure G_i bounded by the arc $\smile M_iM_{i+2}$ and the segment M_iM_{i+2} is not convex for $i = \overline{0, n-1}$.

Such a set of points $P = \{M_i, i = \overline{0, n-1}\}$ is called the *canonical partition* of the boundary of the figure, and the figure $\Phi \in S^{(1)}$ itself is called a *figure with a convex-concave boundary.* The collection of all such figures will be denoted as S. Moreover, the arc $\smile M_iM_{i+1} \subset \Gamma(\Phi)$ will be called *convex* if there exists a segment PQ with ends on the arc $\smile M_iM_{i+1}$ that lies entirely inside the figure Φ; otherwise, it is called *concave* [1, 2].

Thus, if $\Phi \in S$, then the boundary $\Gamma(\Phi)$ is divided into convex and concave sections by the points of the canonical partition. If the arcs $\smile M_{i-1}M_i$ and $\smile M_iM_{i+1}$ have different convexity characteristics, then the point M_i is called *an inflection point* of the boundary of the figure Φ.

The properties of the canonical partition of the figure $\Phi \in S$ significantly depend on the structure of the boundary of the figure Φ. We divide all these figures into three subsets. The subset S_1 includes all figures whose boundaries do not contain such two points $A, B \in \Gamma(\Phi)$ for which $\smile AB$ coincides with the segment AB. The subset S_3 includes all polygons, and the remaining figures form the set S_2, meaning the boundaries of these figures contain at least one straight segment. We define some interesting properties of the canonical partition, some of which significantly depend on the membership of a figure in a particular subset, while others are true for all figures of the class S. When formulating properties, we will indicate for which subsets they are valid.

3 Generalized Tangent for Figures of Class S

In this context, the concept of a figure with a convex-concave boundary without specifying the canonical partition will be incorrect. Therefore, henceforth, we will use the term "figure Φ of class S" or "$\Phi \in S$" to denote not only the figure Φ but also some specific canonical partition $P = \{M_0, M_1, \ldots, M_{n-1}\}$ of its boundary $\Gamma(\Phi)$.

For any point $M \in \Gamma(\Phi)$, we define the concept of a generalized tangent to the figure $\Phi \in S$ passing through the point M. Let $M \notin P$, then $M \in \smile M_iM_{i+1}$. Choose such two arcs $\smile DM$ and $\smile MC$ so that $\smile DC \subset \smile M_iM_{i+1}$. Due to the finiteness of the set of points of the canonical partition, this can always be done. If $M = M_i$ for some $i = \overline{0, n-1}$, then choose such two arcs $\smile DM$ and $\smile MC$ so that $\smile DM \subset \smile M_{i-1}M_i$ and $\smile MC \subset \smile M_iM_{i+1}$. Construct any sequence of points $(\smile D_k) \subset \smile DM$ so that it monotonically converges to the point M monotonically when moving along $\smile DM$ as $k \to \infty$. Then, due to the convexity of the arc $\smile DM$, the sequence of lines D_kM converges to a line which we denote by l_1. Similarly, we find the line l_2 by approaching the sequence of points $(\smile C_k) \subset \smile MC$ to the point M as the limit of the lines C_kM.

The line l passing through the point $M \in \Gamma(\Phi)$ is called *the generalized tangent (GT)* to the figure Φ at the point M (or to the curve $\Gamma(\Phi)$) if one of the following three conditions is met: 1. $\exists\delta > 0$: the segment of the line l with length δ and midpoint at the point M is completely within the figure Φ; 2. $\exists\delta > 0$: the segment of the line l with length δ and midpoint at the point M has no common interior points with the figure Φ; 3. The line l coincides with one of the two lines l_1 or l_2.

4 Definition of the Metric of Tangents

Let figures Φ_1 and Φ_2 from class S be given. Choose any point $M_1 \in \Gamma(\Phi_1)$ and consider some GT l_1 to the figure Φ_1 at the point M_1. Let $l_2^{(1)}, l_2^{(2)}, \ldots, l_2^{(m)}$ be all GTs to the figure Φ_2 that are parallel to the line l_1, and among them, let $l_2^{(j)}$ be the closest to the line l_1 in the sense of the natural distance between parallel lines, which we denote as $\rho(l_1, l_2^{(j)})$. Then, we define

$$r_1(M_1, l_1) = \max_{k=1,\ldots,m} \rho(l_1, l_2^k) = \rho(l_1, l_2^j)$$

$$r_1(M_1) = \max_{l_1 \ni M_1} r_1(M_1, l_1)$$

$$r_1 = \max_{M_1 \in \Gamma(\Phi_1)} r_1(M_1)$$

Similarly, by switching the figures Φ_1 and Φ_2, we define the value r_2.

Definition 1 *The metric of tangents in the class of figures S is called the value*

$$\rho_1(\Phi_1, \Phi_2) = \max\{r_1, r_2\}$$

Theorem 1. *The functional $\rho_1(\Phi_1, \Phi_2)$ is a metric.*

5 Conclusion

In this paper, the metric of tangents was defined and the class of figures in the plane to which it can be appropriately applied was identified.

The metric of tangents possesses a significant property: the convergence of a sequence of figures within a certain class in this metric implies the convergence of their perimeters. However, this metric has been found to be incomplete. Therefore, further research should focus on addressing this incompleteness and exploring the applications of this metric. Specifically, developing efficient algorithms for computing this metric, particularly for polygons, is a key area for future work.

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Groupoids and Quasigroups Obeying Certain Laws

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Artiom Pîrlog

Abstract

In this paper we study under which conditions the binary topological groupoid with some algebraic properties can be "transformed" into a topological quasigroup with the same algebraic properties.

Keywords: *AG*-groupoid, *AD*-groupoid, Ward quasigroup, Schröder quasigroup, Stein's quasigroup, topological quasigroup.

1 Introduction

The results established are related to the results of M. Choban and L. Chiriac in [2] and to the research papers [1,3,4,5,6]. Our main goal is to prove a new connection between groupoids and quasigroups. It is shown that if P is an open compact neighborhood of an $(2,1)$-identity e of an $Ward$ quasigroups G, then P contains an open compact $Ward$ subquasigroup Q with $(2,1)$-identity e. We study properties of AG- and AD-topological quasigroups with multiple identities.

2 Basic Notions

A non-empty set G is said to be a *groupoid* relatively to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a,b) of elements of G there is a unique element $ab \in G$.

A groupoid $(G, \cdot)$ is called a *quasigroup* if for every $a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions.

A groupoid $(G, \cdot)$ is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid $(G, \cdot)$ is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is said to be *subtractive*, if the following conditions hold: $b \cdot (b \cdot a) = a$ and $a \cdot bc = c \cdot ba$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *AD-groupoid* if it satisfies the law $a \cdot bc = c \cdot ba$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

We define a *Ward groupoid* as any groupoid $(G, \cdot)$ containing an element $e \in G$ such that $a^2 = a \cdot a = e$ and $(a \cdot b) \cdot c = a \cdot (c \cdot (e \cdot b))$, for all $a, b, c \in G$. A groupoid $(G, \cdot)$ is called a *Schröder Second Law groupoid* if it satisfies the law $(ab) \cdot (ba) = a$ for all $a, b \in G$ [6]. The identity $(ab) \cdot (ba) = b$ for all $a, b \in G$ is known as *Stein's Third Law* [6]. The concept of (n, m)-identities was introduced by M.M. Choban and L.L. Chiriac in [3].

3 Main Results

We study the problem formulated below from several perspectives.

Problem. Under which conditions the binary topological groupoid with the algebraic properties $P_1, P_2, ..., P_k$ can be "transformed" into a topological quasigroup with the algebraic properties $P_1, P_2, ..., P_k$?

Theorem 1. *If $(G, \cdot)$ is an AG and AD-multiplicative topological groupoids and the following conditions hold:*

1. $x^2 = x \cdot x = x$ for every $x \in G$,

2. $(xy) \cdot (yx) = x$ for all $x, y \in G$.

3. if $xa = ya$ then $x = y$ for all $x, y, a \in G$,

then $(G, \cdot)$ is a Schröder, medial, AG and AD-topological quasigroups.

Example 1. *Let* $G = \{1, 2, 3, 4\}$. *We define the binary operation* "$(\circ)$"*:*

$(\circ)$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

Then $(G, \circ)$ *is a Schröder, AG and AD-non-associative quasigroup.*

Theorem 2. *If* $(G, \cdot)$ *is an AD-multiplicative topological groupoids and the following conditions hold:*

1. $x^2 = x \cdot x = x$ *for every* $x \in G$,

2. $(xy) \cdot (yx) = y$ *for all* $x, y \in G$.

3. if $xa = ya$ *then* $x = y$ *for all* $x, y, a \in G$,

then $(G, \cdot)$ *is a Stein and AD-topological quasigroup.*

Example 2. *Let* $G = \{1, 2, 3, 4, 5\}$. *We define the binary operation* "$(\circ)$"*:*

$(\circ)$	1	2	3	4	5
1	1	4	5	2	3
2	3	2	1	5	4
3	4	5	3	1	2
4	5	3	2	4	1
5	2	1	4	3	5

Then $(G, \circ)$ *is a Stein and AD-non-associative quasigroup.*

Theorem 3. *If* $(G, \cdot)$ *is an AD-multiplicative topological groupoid,* $e \in G$ *and the following conditions hold:*

1. $xe = x$ *for every* $x \in G$,

2. $x^2 = x \cdot x = e$ *for every* $x \in G$,

3. if $xa = ya$ *then* $x = y$ *for all* $x, y, a \in G$,

then $(G, \cdot)$ *is a Ward, subtractive and AD-topological quasigroup with a* $(2, 1)$*-identity* e.

Theorem 4. *Let* $(G, \cdot)$ *be a Ward topological quasigroup with an* $(2, 1)$*-identity* e *and* $x^2 = e$ *for every* $x \in G$. *If* P *is an open compact neighborhood such that* $e \in P$, *then* P *contains an open compact Ward subquasigroup* $(Q, \cdot)$ *with an* $(2, 1)$*-identity of* $(G, \cdot)$.

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On a Method of Prolongation of Quasigroups

Elena Cuzneţov

Abstract

We consider a method of prolongation of finite quasigroups using two transversals, which intersect exactly in one cell, and study the recursive 1-differentiability of such prolongations.

Keywords: quasigroup, transversal, prolongation, recursive derivative, recursively differentiable quasigroup.

The construction of prolongations was first studied by Bruck (1944) who considered finite idempotent quasigroups for this purpose [1]. Note that the notion "prolongation" was introduced by Belousov in 1967 [2].

Constructions of prolongations of finite quasigroups has been given by Osborn (1961), Yamamoto (1961), Denes and Pasztor (1963), Belousov and Belyavskaya (1968), Belyavskaya (1969), Deriyenko and Dudek (2008, 2013) and others.

The recursive derivative of order k of a quasigroup $(Q, \cdot)$, denoted by $(\overset{k}{\cdot})$, where $k \in N$, is defined as follows:

$$x \overset{0}{\cdot} y = x \cdot y;$$
$$x \overset{1}{\cdot} y = y \cdot (x \cdot y);$$
$$x \overset{k}{\cdot} y = (x \overset{k-2}{\cdot} y) \cdot (x \overset{k-1}{\cdot} y), for\ all\ k \geq 2.$$

A binary quasigroup $(Q, \cdot)$ is called recursively r-differentiable if its recursive derivatives $(\overset{k}{\cdot})$, where $k = 1, ..., r$, are quasigroup operations.

It is an open problem to find the maximum order of recursive differentiability of a finite binary (or n-ary) quasigroup. Some estimations are given in [3]–[6], where, in particular, it is shown that:

1. The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q-2$;
2. There exist finite binary recursively $(q-2)$ - differentiable quasigroups of any primary order $q \geq 3$.

The recursive differentiability of some known prolongations is studied in [7]. We give a new method of prolongation and investigate the possibility to extend the recursive differentiability of a finite binary quasigroup to its prolongations in the present work.

In the given method we use 2 transversals having exactly 1 cell in their intersection. The idea of the proposed method of prolongations is shown in the following example, where we give the prolongation of a quasigroup of order 5, with the transversals marked by colored and, respectively, uncolored squares, which intersect in the cell $(1,1)$, with 2 new elements denoted by ξ_1 and ξ_2.

$\cdot$	1	2	3	4	5
1	2	4	1	3	5
2	1	3	5	2	4
3	5	2	4	1	3
4	4	1	3	5	2
5	3	5	2	4	1

$\longrightarrow$

$\circ$	1	2	3	4	5	ξ_1	ξ_2
1	ξ_1	4	1	3	5	2	ξ_2
2	1	ξ_1	ξ_2	2	4	3	5
3	5	2	ξ_1	1	ξ_2	4	3
4	4	ξ_2	3	ξ_1	2	5	1
5	3	5	2	ξ_2	ξ_1	1	4
ξ_1	ξ_2	1	5	4	3	ξ_1	2
ξ_2	2	3	4	5	1	ξ_2	ξ_1

Proposition 1. *For 2 fixed transversals of a finite quasigroup $(Q,\cdot)$, that intersect exactly in one cell, there exist 12 possible prolongations $(Q',\circ)$, with two new elements $\xi_1$and ξ_2, given bellow in I-XII. Moreover, the prolongations I, IV, VI, VII, X and XI are not recursively 1-differentiable, as it is shown after their Cayley tables.*

I.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	a	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_2	ξ_1
ξ_2	...	...	ξ_2	...	...	ξ_1	a

$\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_1$
$\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_1$

II.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	a	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	ξ_1	a
ξ_2	...	...	a	...	...	ξ_2	ξ_1

III.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	ξ_2	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	a	ξ_1
ξ_2	...	...	a	...	...	ξ_1	ξ_2

IV.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_1	...	...	ξ_2	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_1	ξ_2
ξ_2	...	...	ξ_2	...	...	a	ξ_1

$\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_1$
$\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_1$

V.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	a	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_1	ξ_2
ξ_2	...	...	ξ_1	...	...	ξ_2	a

VI.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	a	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	ξ_2	a
ξ_2	...	...	a	...	...	ξ_1	ξ_2

$\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_2$
$\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_2$

VII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	ξ_1	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	a	ξ_2
ξ_2	...	...	a	...	...	ξ_2	ξ_1

$\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_2$
$\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_2$

VIII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	ξ_2	...	...	ξ_1	a
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	a	...	...	ξ_2	ξ_1
ξ_2	...	...	ξ_1	...	...	a	ξ_2

IX.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	ξ_2	a
ξ_2	...	...	ξ_2	...	...	a	ξ_1

X.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	a	ξ_1
ξ_2	...	...	ξ_1	...	...	ξ_2	a

$x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_1 = a$
$x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_2 = a$

XI.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_2	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_2	...	...	ξ_1	a
ξ_2	...	...	ξ_1	...	...	a	ξ_2

$x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_2 = a$
$x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_1 = a$

XII.

$\circ$	...	...	y_0	...	...	ξ_1	ξ_2
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
x_0	...	...	a	...	...	ξ_2	ξ_1
...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...
ξ_1	...	...	ξ_1	...	...	a	ξ_2
ξ_2	...	...	ξ_2	...	...	ξ_1	a

We study if the prolongations of a finite recursively 1-differentiable quasigroup, using 2 transversals which intersect exactly in one cell is recursively 1-differentiable. According to Proposition 1, it remains to consider a total of 6 possible cases.

Proposition 2. *There exist exactly 240 latin squares of order 5, having 2 transversals which intersect exactly in one cell, one of which is on the main diagonal (fixed).*

Proposition 3. *There does not exist recursively 1-differentiable prolongations of quasigroups of order 5, obtained using the main diagonal with the fixed order* $\{2, 3, 4, 5, 1\}$ *and an arbitrary second transversal which intersects the main diagonal exactly in one cell.*

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Semisymmetric Quasigroups

Natalia N. Didurik, Victor A. Shcherbacov

Abstract

A quasigroup satisfying the identity $x(yx) = y$ is called semisymmetric. In this article, the isotopy of semisimmetric quasigroups is studied. A condition is found when a loop, isotopic to a semisymmetric quasigroup, is a semisymmetric loop.

Keywords: Quasigroup, loop, isotopy, semisymmetric quasigroups.

1 Introduction

As a class, the semisymmetric quasigroups arguably warrant particular interest due to both their algebraic and combinatorial properties. Commutatively semisymmetric, i.e., totally symmetric quasigroups, have been studied for almost as long as quasigroups themselves [1].

In [2], a bijection between totally symmetric quasigroups and directed graphs meeting certain specifications is established. Alex W. Nowak in [3] studied modules over semisymmetric quasigroups.

2 Preliminaries

Definition 1. A binary groupoid $(Q, \cdot)$ is called a quasigroup, if for any ordered pair $(a, b) \in Q^2$, there exist unique solutions $x, y \in Q$ to the equations $x \cdot a = b$ and $a \cdot y = b$ ([4, 5].

Definition 2. A quasigroup $(K, \cdot)$ is called semisymmetric if it satisfies the semisymmetric law:

$$x(yx) = y, \tag{1}$$

$\forall x, y \in K$ [5].

3 Results

Theorem 1. In any semisymmetric quasigroup $(K,\cdot)$, the following holds:

a) $e_x = f_x$,$\forall x \in K$, where $f_x x = x e_x = x$.

b) $x \cdot yx = xy \cdot x$, $L_x R_x = R_x L_x$, $L_x = R_x^{-1}$, $R_x = L_x^{-1}$, $xx = f_x = e_x, \forall x, y \in K$.

c) The equations $xa = b$ and $ay = b$ have, respectively, the solutions $x = ab$, $y = ba$.

Theorem 2. A quasigroup $(K,\circ)$, isotopic to a semisymmetric quasigroup $(K,\cdot)$, where $x \circ y = \gamma^{(-1)}(\alpha x \cdot \beta y)$ is a semisymmetric quasigroup if and only if $T = (\gamma\alpha^{-1}, \alpha\beta^{-1}, \beta\gamma^{-1})$ is an autotopy of the quasigroup $(K,\cdot)$.

Example 1. We define the semisymmetric quasigroup $(K,\cdot)$ using the Cayley table. Let $K = \{1,2,3,4\}$.

$\cdot$	1	2	3	4
1	1	4	2	3
2	3	2	4	1
3	4	1	3	2
4	2	3	1	4

In the quasigroup $(K,\cdot)$, the following holds: 1) $x \cdot xy = y$, 2) $x(yz \cdot t) = t(xz \cdot y)$, 3) $x^2 = x$, 4) $x \cdot xy = yx$, $yx \cdot x = xy$ (left and right Stein identities), 5) $x \cdot yz = xy \cdot xz$, $yz \cdot x = yx \cdot zx$ (distributive laws), 6) $x \cdot yx = xy \cdot x$ (elastic law), 7) $xy \cdot uv = xu \cdot yv$ (medial law). These examples show the way in which semisymmetric quasigroups can be studied.

Example 2. We define a loop $(K,\circ)$, isotopic to the quasigroup $(K,\cdot)$, where the isotopy is of the form $x \circ y = R_4^{-1}x \cdot L_2^{-)}y$. $(K,\circ)$ is an abelian group, $x \circ x = 1$.

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Theorem 3. A loop $(K, \circ)$, isotopic to a semisymmetric quasigroup $(K, \cdot)$, where the isotopy is given by the equality $x \circ y = R_a^{-1}x \cdot L_b^{-1}y$, $\forall x, y, a, b \in K$, a, b – are any fixed elements, will be a semisymmetric loop if and only if, $\forall x, y \in K$, the equality

$$b(xa \cdot y) = x(ya \cdot b), \tag{2}$$

holds in $(K, \cdot)$.

Then, naturally, the problem arises to study a semisymmetric quasigroup $(K, \cdot)$ with the identity

$$x \cdot (yz \cdot t) = t \cdot (xz \cdot y), \tag{3}$$

$\forall x, y, z, t \in K$.

The semisymmetric quasigroup $(K, \circ)$ from Example 2, satisfies the identity 3.

Theorem 4. A semisymmetric quasigroup $(K, \cdot)$, in which the identity 3 is satisfied, is isotopic to the abelian group $(K, \circ)$, where $x \circ x = e$, $\forall x \in K$, the element e is the identity element of the group $(K, \circ)$, and the isotopy is of the form:

$$x \circ y = R_a^{-1} \cdot L_b^{-1}y, \tag{4}$$

$\forall x, y \in K$.

4 Conclusion

In this paper, some properties of semisymmetric quasigroups are studied.

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On the Volumes of Polyhedra which Tile the Hyperbolic Space Face-to-Face and Solid-Transitively

Ion Gutsul

Abstract

We constructed five countable series of bounded and unbounded polyhedra of the same volume in H^3.

Keywords: Polyhedra, volume, hyperbolic space, synthetic geometry.

In [1], Milnor introduced the function $\Lambda(\alpha) = -\int_0^\alpha \ln|2\sin u|du$ which he used for calculating the volumes of hyperbolic polyhedra. He showed that the volume of a simplex with all infinitely remote vertices is equal to $V(\Delta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$, where α, β, γ are dihedral angles of the simplex which satisfy the equality $\alpha + \beta + \gamma = \pi$.

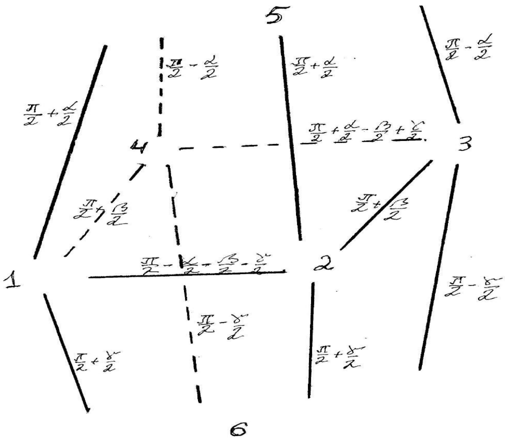
Later on, in many works, the questions on the volumes of fundamental polyhedra of discrete motion groups of the hyperbolic space arose.

In the present communication, we state the following.

Theorem 1. *In the hyperbolic space H^3, there exist five countable series of bounded and unbounded polyhedra of the same volume which tile the space face-to-face and solid-transitively and possess the following properties: dihedral angles of polyhedra depend on three integer parameters; for a fixed choice of parameters, there exist at least five polyhedra with the same volume such that one polyhedron is bounded, three polyhedra are unbounded with one infinitely remote vertex but have different dihedral angles, and one polyhedron has three infinitely remote vertices.*

We prove this theorem using synthetic geometry of hyperbolic space H^3; moreover, we construct the mentioned polyhedra. The construction is done for continuous parameters (i.e., angles of polyhedra); however, we are interested only in polyhedra that tile H^3 face-to-face and solid-transitively.

Let the angles α, β, γ satisfy the conditions: $\alpha+\beta+\gamma < \pi, \alpha > \beta > \gamma$. Consider octahedron O_1 with all infinitely remote vertices depicted in Fig. 1.



O_1
Fig. 1

We denote its infinitely remote vertices by numbers $1, 2, 3, 4, 5, 6$. It is easy to show that such an octahedron exists and to obtain its partition into four simplices with all infinitely remote vertices. To the octahedron O_1 we "glue" along face $(1, 2, 5)$ a simplex with all infinitely remote vertices and dihedral angles α, $\pi\backslash 2$ - $\alpha\backslash 2$, $\pi\backslash 2 - \alpha\backslash 2$, along faces $(1, 4, 6)$ and $(2, 3, 6)$ quadrangular pyramids with all infinitely remote vertices. Dihedral angles at the bases of pyramids are equal to: $\beta\backslash 2 + \gamma\backslash 2$ at $(1, 7)$; $(2, 9)$, $\beta\backslash 2$ - $\gamma\backslash 2$ at $(4, 8)$, $(3, 10)$ and $\pi\backslash 2$ - $\beta\backslash 2$ at $(1, 4)$ and $(7, 8)$. So, we obtain polyhedron R_1 with all infinitely remote vertices depicted in Fig. 2.

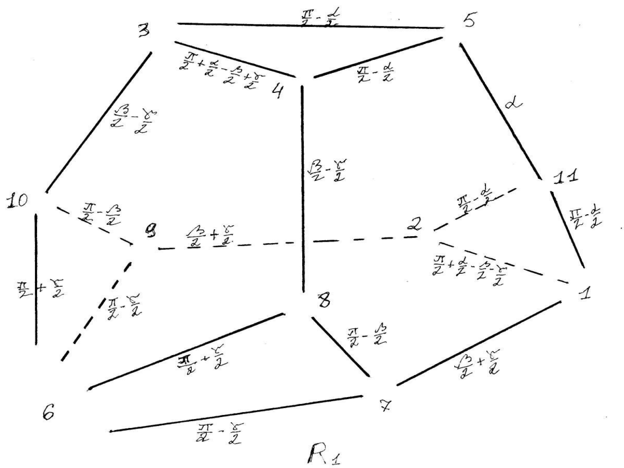


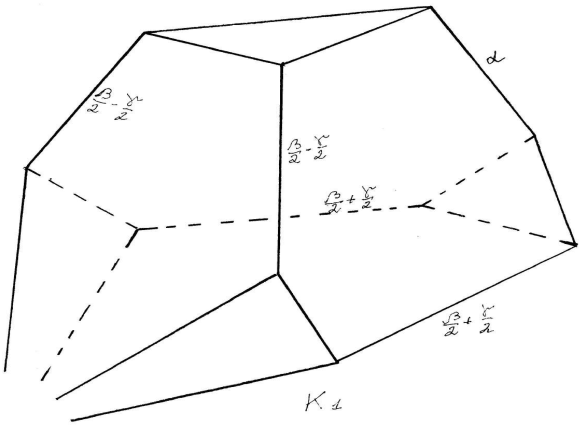
Fig. 2

Edges $(3,10)$, $(4,8)$, and $(5,11)$ of this polyhedron form a hyperbolic bundle of straight lines; therefore, there exists a plane orthogonal to these edges. An orthogonal plane exists analogously for edges $(1,7)$, $(2,9)$, and $(11,5)$. Also, for quadrangular pyramids with infinitely remote vertices $(6,7,1,4,8)$ and $(6,10,3,2,9)$ there exist symmetry planes which are orthogonal to edges $(1,7)$ and $(4,8)$, respectively, edges $(3,10)$ and $(2,9)$. After cutting from polyhedron R_1 some parts with above planes, we get polyhedron K_1 depicted in Fig. 3.

This polyhedron has only one infinitely remote vertex, the other vertices are proper. All the dihedral angles which are not indicated in Fig. 3 are right, i.e., are equal to $\pi\backslash 2$.

Using the same reasoning, we get two more polyhedra K_2 and K_3 which have analogous combinatorial type. But dihedral angles at analogous edges are equal to $(\beta, \alpha\backslash 2 - \gamma\backslash 2, \alpha\backslash 2 + \gamma\backslash 2)$ in one case and to $(\gamma, \alpha\backslash 2 - \beta\backslash 2, \alpha\backslash 2 + \beta\backslash 2)$ in other case. All the rest angles are right.

By the same way, we get two more polyhedra, one of them is



K 1

Fig. 3

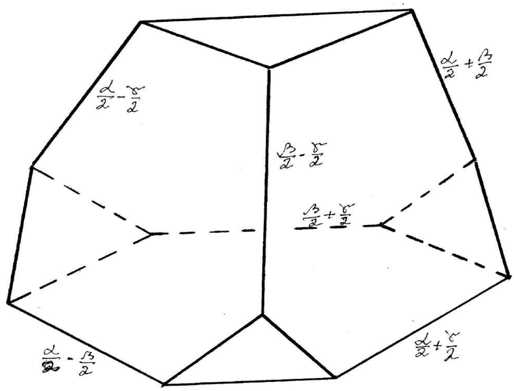
bounded (see Fig.4) and the other is unbounded with three infinitely remote vertices (see Fig.5).

All the dihedral angles without labels of polyhedra K_4 and K_5 are equal to $\pi\backslash 2$.

The volumes of all polyhedra K_1, K_2, K_3, K_4, and K_5 are equal to:
$V(K_i) = \Lambda(\pi\backslash 4 - \alpha\backslash 4 - \beta\backslash 4 - \gamma\backslash 4) + \Lambda(\pi\backslash 4 + \alpha\backslash 4 + \beta\backslash 4 + \gamma\backslash 4) +$
$+ \Lambda(\pi\backslash 4 - \alpha\backslash 4 - \beta\backslash 4 + \gamma\backslash 4) + \Lambda(\pi\backslash 4 + \alpha\backslash 4 + \beta\backslash 4 - \gamma\backslash 4) +$
$+\Lambda(\pi\backslash 4 - \alpha\backslash 4 + \beta\backslash 4 - \gamma\backslash 4) + \Lambda(\pi\backslash 4 + \alpha\backslash 4 - \beta\backslash 4 + \gamma\backslash 4) +$
$+ \Lambda(\pi\backslash 4 + \alpha\backslash 4 - \beta\backslash 4 - \gamma\backslash 4) + \Lambda(\pi\backslash 4 - \alpha\backslash 4 + \beta\backslash 4 + \gamma\backslash 4)$

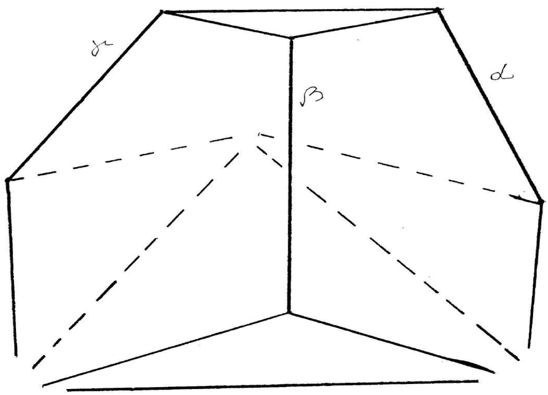
In order that constructed polyhedra would tile H_3 face-to-face and solid-transitively, it is sufficient to make them Coxeter.

Let $k, n, m \in N$ and $k > n > m$, we take angles:
$\alpha = \pi\backslash(m(m+n)(n-m)(m+k)(k-m)(n+k)(k-n))$
$\beta = \pi\backslash(n(m+n)(n-m)(m+k)(k-m)(n+k)(k-n))$
$\gamma = \pi\backslash(k(m+n)(n-m)(m+k)(k-m)(n+k)(k-n))$,
and then any polyhedron K_1, K_2, K_3, K_4, K_5 tiles H_3 face-to-face and solid-transitively. Therefore, we have a countable series of such poly-



K_4

Fig. 4



K_5

Fig. 5

hedra which depends on three integer parameters.

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About AC-Groupoids

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Abstract

The concept of a right (left) AC-groupoid over an arbitrary group is defined and studied. Necessary and sufficient conditions which transform a right (left) AC-groupoid into a quasigroup are given. The AC-groupoids form a wide class of special groupoids and quasigroups that have a transitive subset of automorphisms.

Keywords: groupoid, automorphism, quasigroup, group, AC-groupoid.

1 Introduction

The concept of "automorphic by the cyclic group" groupoid (AC-groupoid) was introduced by A. Sade [2]. The author considers the problem of defining a single-valued binary operation ($\times$) on a cyclic group G, such that the mapping $x \to x + t$ is an automorphism of $(G, \times)$ for each t in G. Quasigroups of finite order n, automorphic by the cyclic group, exist for every odd n, does not exist for even n and, could not be groups. Based on the idea of A. Sade [2], we introduse the concepts of right (left) AC-groupoids over an arbitrary group, generalizing the concept of groupoid "automorphic by the cyclic group". The properties of right (left) AC-groupoids are investigated in the present paper, and as a result, a wide class of groupoids with the transitive automorphism group is obtained. We find the conditions when the AC-groupoid is a quasigroup (either idempotent or comutative one). A right AC-quasigroup is constructed over the Frobenius group of order 21 (the smallest non-Abelian group of odd order). Note that AC-quasigroups are of interest for cryptography.

2 Right (left) AC-groupoids

Definition 1. *A groupoid $(Q, *)$ is called a right AC-groupoid (respectively a left AC-groupoid), if there exists a group $(Q, +)$ such that the right translations*

$$R_a^+ : R_a^+(x) = x + a$$

(respectively left translations $L_a^+ : L_a^+(x) = a + x$) of the group $(Q, +)$ are automorphisms of $(Q, *)$.

Theorem 2. *A groupoid $(Q, *)$ is a right AC-groupoid (respectively, left AC-groupoid) over a group $(Q, +)$, if and only if for any $x, y \in Q$ we have*

$$x * y = f(x - y) + y \tag{1}$$

(*respectively,*

$$x * y = x + g(-x + y)), \tag{2}$$

where $f(x) = x * 0$ (respectively, $g(x) = 0 * x$), 0 is the neutral element of the group $(Q, +)$ and " $-$ " is the inverse operation of " $+$ ".

Proof. Let $(Q, *)$ be a right AC-groupoid over the group $(Q, +)$. As $R_a^+ \in Aut(Q, *)$ for any $a \in Q$, we have

$$x * a = [(x - a) + a] * [(a - a) + a] = [(x - a) + a] * [0 + a] =$$

$$= [(x - a) * 0] + a = f(x - a) + a,$$

for every $x \in Q$. Thus, $x * y = f(x - y) + y$, $\forall x, y \in Q$.

Conversely, if $(Q, +)$ is a group, then $(Q, *)$, defined in (1), is a right AC-groupoid over the group $(Q, +)$. Indeed, for any $a \in Q$ we have

$$(x*y)+a = [f(x-y)+y]+a = f(x-y)+(y+a) = f(x+a-a-y)+(y+a) =$$

$$= f((x + a) - (y + a) + (y + a) = (x + a) * (y + a).$$

So $R_a^+ \in Aut(Q, *)$ for any $a \in (Q, *)$. Thus, the right AC-groupoid $(Q, *)$ is determined by the mapping $f : Q \to Q, f(x) = x * 0$ and the group $(Q,+)$ with the neutral element 0, by the formula $x * y = f(x-y)+y$, for any $x, y \in Q$. We will denote this groupoid by $(Q, *_f)$ or $(Q, *_f, +, 0)$. For a left AC-groupoid the proof is similar and we will denote the corresponding groupoid by $(Q, {}_g*)$ or $(Q, {}_g*, +, 0)$. □

So

$$x *_f y = f(x-y)+y \tag{3}$$

$$x {}_g* y = x + g(-x+y) \tag{4}$$

for any $x, y \in Q$.

It is clear that two binary operations Δ and ∇ defined on the same set Q are different if and only if there are elements $m, n \in Q$ such that $m\Delta n \neq m\nabla n$.

Theorem 3. *There is a one-to-one correspondence between the set of right AC-groupoids defined over a group $(Q,+)$ and the set $F(Q)$ of all mappings defined on the set Q. For $|Q| = n$ there are exactly n^n right AC-groupoids defined over the group $(Q,+)$.*

Similar statements are true for left AC-groupoids as well.

Theorem 4. *There is a one-to-one correspondence between the set of the left AC-groupoids defined on a group $(Q,+)$ and the set $F(Q)$ of all mappings defined on the set Q. For $|Q| = n$ there are exactly n^n left AC-groupoids defined over the group $(Q,+)$.*

3 Some properties of right (left) AC-groupoids

Proposition 5. *In any right (left) AC-groupoid, defined over a group $(Q,+)$, the mapping $x \to x * x$ is a bijection.*

Remark 6. *In the Cayley table of a finite right (left) AC-groupoid all elements on the main diagonal are different.*

Proposition 7. *An automorphism φ of the group $(Q,+)$ is an automorphism of the right AC-groupoid $(Q, *_f)$ (left AC-groupoid $(Q, {}_g*)$) if and only if φ and f (respectively φ and g) are commutative, that is, for any $x \in Q$, $\varphi(f(x)) = f(\varphi(x))$ (respectively $\varphi(g(x)) = g(\varphi(x))$).*

Proposition 8. *Let* $(Q, *_f)$ ($(Q,_g *)$) *be a right (left) AC-groupoid defined over a group* $(Q, +)$. *Then* $g(x) = f(-x)+x$, $(f(x) = x+g(-x))$ *for any* $x \in Q$.

Theorem 9. *A right AC-groupoid* $(Q, *_f)$ *(left AC-groupoid* $(Q,_g *)$*), defined over the group* $(Q, +)$ *with the neutral element* 0 *is commutative if and only if* $x *_f 0 = 0 *_f x$ *for any* $x \in Q$ *(respectively,* $0_g * x = x_g * 0$ *for any* $x \in Q$*).*

Proposition 10. *If the group* $(Q, +)$ *has an element of order two, then any right AC-groupoid* $(Q, *_f)$ *(respectively left AC-groupoid* $(Q,_g *)$*) is not commutative.*

Corollary 11. *Any right AC-groupoid* $(Q, *_f)$ *(left AC-groupoid* $(Q,_g *)$*) of even order is not commutative.*

Theorem 12. *A right AC-groupoid* $(Q, *_f)$ *(left AC-groupoid* $(Q,_g *)$*), defined over the group* $(Q, +)$ *with the neutral element* 0 *is idempotent if and only if* $f(0) = 0$ *(respectively* $g(0) = 0$*).*

4 Right (left) *AC*-quasigroups

Theorem 13. *Let* $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *be a right (left) AC-groupoid defined over the group* $(Q, +)$. *Then:*

1. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a right-reducible groupoid if and only if the mapping* $f(x) = x * 0$ *is injective;*

2. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a groupoid with right divisibility if and only if the mapping* $f(x) = x * 0$ *is surjective;*

3. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a right quasigroup if and only if the the mapping* $f(x) = x * 0$ *is bijective.*

4. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a groupoid with left reduction if and only if the mapping* $g(x) = 0 * x$ *is injective;*

5. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a groupoid with left divisibility if and only if the mapping* $g(x) = 0 * x$ *is surjective;*

6. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a left quasigroup if and only if the the mapping* $g(x) = 0 * x$ *is bijective.*

7. $(Q, *_f, +, 0)$ ($(Q,_g *, +, 0)$) *is a quasigroup if and only if the mappings* $g(x) = 0 * x$ *and* $f(x) = x * 0$ *are bijective.*

Figure 2 contains a right $AC-$quasigroup constructed over the Frobenius group of order 21 given in Figure 1.

°	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	0	13	14	12	4	5	3	16	17	15	7	8	6	19	20	18	10	11	9
2	2	0	1	8	6	7	14	12	13	20	18	19	5	3	4	11	9	10	17	15	16
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2
4	4	5	3	16	17	15	7	8	6	19	20	18	10	11	9	1	2	0	13	14	12
5	5	3	4	11	9	10	17	15	16	2	0	1	8	6	7	14	12	13	20	18	19
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
7	7	8	6	19	20	18	10	11	9	1	2	0	13	14	12	4	5	3	16	17	15
8	8	6	7	14	12	13	20	18	19	5	3	4	11	9	10	17	15	16	2	0	1
9	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8
10	10	11	9	1	2	0	13	14	12	4	5	3	16	17	15	7	8	6	19	20	18
11	11	9	10	17	15	16	2	0	1	8	6	7	14	12	13	20	18	19	5	3	4
12	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	12	4	5	3	16	17	15	7	8	6	19	20	18	10	11	9	1	2	0
14	14	12	13	20	18	19	5	3	4	11	9	10	17	15	16	2	0	1	8	6	7
15	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
16	16	17	15	7	8	6	19	20	18	10	11	9	1	2	0	13	14	12	4	5	3
17	17	15	16	2	0	1	8	6	7	14	12	13	20	18	19	5	3	4	11	9	10
18	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
19	19	20	18	10	11	9	1	2	0	13	14	12	4	5	3	16	17	15	7	8	6
20	20	18	19	5	3	4	11	9	10	17	15	16	2	0	1	8	6	7	14	12	13

Figure 1.

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$\times_f$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	2	1	6	17	10	12	11	19	18	5	7	3	20	16	9	14	4	15	8	13
1	2	1	0	11	7	15	20	13	9	8	19	3	17	4	18	5	10	12	14	16	6
2	1	0	2	16	9	8	10	18	14	4	6	20	19	15	5	13	3	11	7	12	17
3	18	11	16	3	5	4	9	20	13	15	14	1	0	8	10	6	2	19	12	17	7
4	17	19	9	5	4	3	14	10	18	2	16	12	11	1	6	20	7	0	8	13	15
5	10	15	20	4	3	5	19	12	11	13	0	17	7	9	2	1	18	8	16	6	14
6	15	20	10	0	14	19	6	8	7	12	2	16	18	17	4	3	11	13	9	5	1
7	11	16	18	20	1	12	8	7	6	17	13	0	5	19	15	14	4	9	2	10	3
8	19	9	17	13	18	2	7	6	8	1	15	14	16	3	20	10	12	5	4	0	11
9	12	8	4	18	2	13	3	17	1	9	11	10	15	5	19	0	20	7	6	14	16
10	5	13	6	14	19	0	2	4	15	11	10	9	20	16	3	8	1	18	17	7	12
11	7	3	14	1	12	20	16	0	5	10	9	11	4	18	17	19	6	2	13	15	8
12	9	17	19	15	11	7	0	5	16	6	20	4	12	14	13	18	8	1	3	2	10
13	20	10	15	8	16	9	17	1	3	5	7	18	14	13	12	2	19	6	11	4	0
14	16	18	11	10	6	17	4	15	2	19	3	8	13	12	14	7	0	20	1	9	5
15	6	15	13	12	20	1	18	14	10	3	8	19	9	2	7	15	17	16	0	11	4
16	14	17	15	7	8	6	19	20	18	10	11	9	1	2	0	13	14	12	4	5	3
17	4	15	16	2	0	1	8	6	7	14	12	13	20	18	19	5	3	4	11	9	10
18	3	14	7	9	8	16	15	2	4	0	17	13	6	11	1	12	5	10	18	20	19
19	8	4	12	17	10	6	5	16	0	14	1	15	2	7	9	11	13	3	20	19	18
20	13	6	5	7	15	11	1	3	17	16	12	2	10	0	8	4	9	14	19	18	20

Figure 2.

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Invariant Transformations of Loop Transversals in Loops. 1. The Case of Isomorphism

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Abstract

A special class of invariant transformations of loop transversals in loops is investigated. The considered transformations correspond to arbitrary isomorphisms of transversal operations in loops. Isomorphisms of loop transversal operations with the same unit 1 are investigated.

Keywords: quasigroups, loops, transversals in loops, invariant transformations, isomorphism.

1 Introduction

The notion of left (right) transversal in a group to its subgroup is well known in group theory, group representation theory, and in quasigroup theory [1, 3]. This notion has been studied during the last 70 years (since R. Baer's work [1]). Loop transversals (transversals whose transversal operations are loops) in some concrete groups to their subgroups are of special interest.

In the work [6], the author gives a natural generalization of the notion of transversal for a suitable class of loops. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloop (see [8] and the **Condition A** below).

Let a loop L and its proper subloop R be set, and some loop transversal $T_0 = \{t_i\}_{i \in E}$ in L to R is given and fixed. How to describe all other loop transversals in L to R? In other words, what kind of transformations are admissible over loop transversal T_0 to get that the obtained sets are loop transversals too? And how to describe the set of all such admissible transformations?

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, they are transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies, and isostrophies are well studied.

In the present work, the author investigates what transformations of loop transversals correspond to the first well-known transformation of transversal operations – to an isomorphism. A similar investigation for loop transversals in groups can be found in [7].

2 Necessary definitions and statements

2.1 Quasigroups, loops, and transversals in loops

Definition 1 . *A system $\langle E, \cdot \rangle$ is called [2] a* ***right (left) quasigroup*** *if, for arbitrary $a, b \in E$, the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If a system $\langle E, \cdot \rangle$ is both a right and left quasigroup, then it is called a* ***quasigroup****. If, in a right (left) quasigroup $\langle E, \cdot \rangle$, there exists an element $e \in E$ such that $x \cdot e = e \cdot x = x$, for every $x \in E$, then the system $\langle E, \cdot \rangle$ is called a* ***right (left) loop*** *(the element e is called a* ***unit*** *or* ***identity element****). If a system $\langle E, \cdot \rangle$ is both a right and left loop, then it is called a* ***loop****.*

Definition 2 . *([8]) Let $\langle L, \cdot \rangle$ be a loop and $\langle R, \cdot \rangle$ be its proper subloop. Then a* ***left coset*** *of R is a set of the form*

$$xR = \{xr \mid r \in R\},$$

*and a **right coset** has the form*

$$Rx = \{rx \mid r \in R\}.$$

Definition 3. *([8]) A loop L has a **left (right) coset decomposition by its proper subloop** R, if the left (right) cosets form a partition of the loop L, i.e., for some set of indexes E,*

1. $\bigcup_{i \in E} (a_i R) = L$;
2. *for every* $i, j \in E, \ i \neq j$

$$(a_i R) \cap (a_j R) = \varnothing.$$

Definition 4. ***(Left Condition A)** A product at the left of an arbitrary element a of a loop L and an arbitrary left coset R_i of a loop L by its proper subloop R is a left coset of the loop L by its proper subloop R too, i.e., for every $a, b \in L$, there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{1}$$

2.2 The concept of a transversal in a loop to its subloop. Elementary properties.

Definition 5. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and the left (right) **Condition A** is fulfilled. Let $\{R_x\}_{x \in E}$ be the set of all left (right) cosets in L to R that form a left (right) coset decomposition of the loop L. The set $T = \{t_x\}_{x \in E} \subset L$ is called a **left (right) transversal** in L to R if T is a complete set of representatives of the left (right) cosets R_x in L to R, i.e., there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.*

Definition 6. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and both left and right **Conditions A** are fulfilled. The set $T = \{t_x\}_{x \in E}$ is called a **two-sided transversal** in L to R, if it is a left and a right transversal in L to R.*

Remark 7. *In Definitions 5, we assume that $t_1 = e$ (for simplicity and analogously to the case of transversals in groups).*

On the set E, it is possible to define correctly the following operation:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \quad (2)$$

if T is the left (non-reduced) transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \quad (3)$$

if T is the right (non-reduced) transversal in L to R.

Further we will study only the left transversals in L to R (the study of the right transversals is similar).

Definition 8. *If a system $< E, \overset{(T)}{\cdot}, 1 >$ is a loop, then the left transversal $T = \{t_x\}_{x \in E}$ is called a **loop transversal**.*

The following statement is known (see [6]):

Lemma 9. *A system $< E, \overset{(T)}{\cdot}, 1 >$ is a left loop with the two-sided unit 1.*

2.3 Representation of loops by cosets

Let $\langle L, \cdot, e \rangle$ be a loop, $\langle R, \cdot, e \rangle$ be its subloop, and the left **Condition A** is fulfilled in $\langle L, \cdot, e \rangle$. Let $T = \{t_x\}_{x \in E}$ be a left transversal in L to R. Define the following mapping:

$$\begin{aligned} f &: L \times E \to E, \\ f &: (g, x) \to y = \hat{g}(x), \\ \hat{g}(x) &= y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot R) = t_y \cdot R. \end{aligned} \quad (4)$$

By virtue of the left **Condition A**, this definition (a *left action* of the loop L on the set E) is correct.

Lemma 10. *The mapping $\hat{g}$ is a permutation on the set E, for every element $g \in L$.*

From the previous Lemma, we obtain a permutation representation of the loop $\langle L, \cdot, e\rangle$: the mapping φ

$$\begin{aligned} \varphi &: g \to \hat{g}, \\ \varphi &: L \to \hat{L} \subset S_E, \end{aligned}$$

(where S_E is the symmetric group on E), and the multiplication of permutations from $\hat{L}$ is defined as follows:

$$\hat{g}_1 * \hat{g}_2 = \hat{g}_3 \quad \overset{def}{\Leftrightarrow} \quad g_1 \cdot g_2 = g_3 \quad \text{in a loop } \langle L, \cdot, e\rangle. \tag{5}$$

It is easy to see that

$$\varphi(g_1) * \varphi(g_2) = \hat{g}_1 * \hat{g}_2 = \hat{g}_3 = \widehat{g_1 \cdot g_2} = \varphi(g_1 \cdot g_2),$$

i.e., φ is a homomorphism of loops, from $\langle L, \cdot, e\rangle$ to $\left\langle \hat{L}, *, \mathbf{id} \right\rangle$.

Lemma 11. *For an arbitrary left transversal $T = \{t_x\}_{x \in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$, the following statements are true:*

1. *$\forall r \in R$: $\hat{r}(1) = 1$;*
2. *$\forall x, y \in E$:*

$$\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_x^{-1}(y) = x \backslash y,$$

where $\hat{t}_x^{-1}$ is the inverse of a permutation $\hat{t}_x$ in S_E, and "$\backslash$" is the left division in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\begin{gathered} \hat{t}_x(1) = x, \quad \hat{t}_1(x) = x, \\ \hat{t}_x^{-1}(1) = x\backslash 1; \quad \hat{t}_x^{-1}(x) = 1. \end{gathered}$$

Let us show how any two left transversals T and P in a loop L to its subloop R are connected.

Lemma 12 (see also [5]). *Let $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$- be left transversals in L to R. Then there exists a set of elements $\{r_{(x)}\}_{x \in E}$ from R such that:*

1. $p_x = t_x r_{(x)}\ \forall x \in E$;
2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{r}^*_{(x)}(y)$, *where* $\hat{r}^*_{(x)}$ *is some suitable permutation from* $\hat{R}$.

This set $\{r_{(x)}\}_{x \in E}$ may be called a **derivation set** for the transversal T (and for the transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

3 The transformations which correspond to isomorphisms of the transversal operations

Let us remind the definitions of a left multiplicative group and of a left inner mappings group of a loop.

Definition 13. *Let $< E, \cdot, e >$ be a loop. Then the group*

$$LM(< E, \cdot, e >) \overset{def}{=} < L_a \mid a \in E >,$$

*generated by all left translations L_a of the loop $< E, \cdot, e >$, is called the **left multiplicative group** of the loop $< E, \cdot, e >$. Its subgroup*

$$LI(< E, \cdot, e >) \overset{def}{=} < l_{a,b} \mid l_{a,b} = L^{-1}_{a \cdot b} L_a L_b, : a, b \in E >$$

*generated by all permutations $l_{a,b}$, is called the **left inner mappings group** of the loop $< E, \cdot, e >$.*

Let $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$ be two loop transversals in a loop L to its subloop R, $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ be its transversal operations. We will fix one of transversals, for example, $T = \{t_x\}_{x\in E}$ and will consider the following group:

$$M_L(T) \stackrel{def}{=} < \alpha \mid \alpha \in St_1(S_E),\ LM(< E, \stackrel{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{LM(L)} \alpha^{-1} >,$$

generated by all permutations $\alpha \in St_1(S_E)$ which satisfy the condition

$$LM(< E, \stackrel{(T)}{\cdot}, 1 >) \subseteq \alpha \widehat{LM(L)} \alpha^{-1}.$$

Lemma 14. *The following statement is true:*

$$N_{St_1(S_E)}(\widehat{LM(L)}) \subseteq M_L(T) \subseteq St_1(S_E).$$

Lemma 15. *Let the loops* $< E, \stackrel{(T)}{\cdot}, 1 >$ *and* $< E, \stackrel{(P)}{\cdot}, 1 >$ *be isomorphic, and let* $\varphi : E \to E$ *be an isomorphism (note that* $\varphi(1) = 1$*). Then*

1. $\widehat{P} = r_0^{-1} \widehat{T} r_0$ *for some* $r_0 \in R^* = M_L(T)$*;*
2. $\varphi \equiv r_0$*.*

Lemma 16. *Let* $T = \{t_x\}_{x\in E}$ *be a fixed loop transversal in* L *to* R*,* $r_0 \in R^* = M_L(T)$ *and let*

$$\hat{p}_{x'} \stackrel{def}{=} r_0^{-1} \hat{t}_x r_0 \quad \forall x \in E.$$

The following statements are true:

1. $P = \{p_{x'}\}_{x'\in E}$ *is a loop transversal in* G *to* H*;*
2. *The transversal operations* $< E, \stackrel{(P)}{\cdot}, 1 >$ *and* $< E, \stackrel{(T)}{\cdot}, 1 >$ *are isomorphic, and the isomorphism is set up by the mapping* $\varphi(x) = r_0(x)$*.*

It means that "conjugated" loop transversals in L to R correspond to isomorphic loop transversal operations and vice versa.

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Invariant Transformations of Loop Transversals in Loops. 2. The Case of Isotopy

Kuznetsov Eugene

Abstract

A special class of invariant transformations of loop transversals in loops is investigated. Transformations from this class correspond to arbitrary isotopies of transversal operations in loops. Isotopies of loop transversal operations with the same unit 1 are investigated.

Keywords: quasigroups, loops, transversals in loops, invariant transformations, isotopy.

1 Introduction

In [6], the author gives a version of a natural generalization of the notion of transversal at a suitable class of loops. As the elements of a left (right) transversal in a group to its subgroup are representatives of every left (right) coset to the subgroup, the notion of a left (right) transversal in a loop to its subloop can be correctly defined only in the case when this loop admits a left (right) coset decomposition by its subloops (see [8] and the **Condition A** below).

Let a loop L and its proper subloop R be set, and $T_0 = \{t_i\}_{i \in E}$ be a fixed loop transversal in L to R. How to describe all other loop transversals in L to R? In other words, what kind of transformations are admissible over a loop transversal T_0 in order to get that the obtained sets are loop transversals too? And how to describe the set of all such admissible transformations?

Generally speaking, such transformations are known, but not for transversals, only for operations – they are isomorphisms, isotopies, parastrophies (of a certain kind), isostrophies (of a certain kind) and crossed isotopies (of a certain kind). But firstly, t hey a re transformations of operations (transversal operations, in particular) instead of transversals; and secondly, only isomorphisms, isotopies, and isostrophies are well studied.

In the present work, the author investigates the transformations of loop transversals that correspond to the well-known transformations of transversal operations – to isotopies. Similar investigation of transformations of loop transversals in groups can be found in [7].

2 Necessary definitions and statements

2.1 Quasigroups, loops, and transversals in loops

Definition 1 . *A system $\langle E, \cdot\rangle$ is called [2] a* ***right (left) quasigroup*** *if, for arbitrary $a, b \in E$, the equation $x \cdot a = b$ ($a \cdot y = b$) has a unique solution in the set E. If a system $\langle E, \cdot\rangle$ is both a right and a left quasigroup, then it is called a* ***quasigroup****. If, in a right (left) quasigroup $\langle E, \cdot\rangle$, there exists an element $e \in E$ such that*

$$x \cdot e = e \cdot x = x,$$

for every $x \in E$, then the system $\langle E, \cdot\rangle$ is called a ***right (left) loop*** *(the element e is called a* ***unit*** *or* ***identity element****). If a system $\langle E, \cdot\rangle$ is both a right and left loop, then it is called a* ***loop****.*

Definition 2 . *([8]) Let $\langle L, \cdot\rangle$ be a loop and $\langle R, \cdot\rangle$ be its proper subloop. Then a* ***left coset*** *of R is a set of the form*

$$xR = \{xr \mid r \in R\},$$

and a ***right coset*** *has the form*

$$Rx = \{rx \mid r \in R\}.$$

Definition 3. *([8]) A loop L has a* ***left (right) coset decomposition by its proper subloop*** *R, if the left (right) cosets form a partition of the loop L, i.e., for some set of indexes E, we have:*

1. $\bigcup\limits_{i\in E}(a_iR) = L$;

2. *for every* $i, j \in E,\ \ i \neq j$

$$(a_iR) \cap (a_jR) = \varnothing.$$

Definition 4. *(**Left Condition A**) The product on the left of an arbitrary element a of a loop L and an arbitrary left coset R_i of a loop L by its proper subloop R is a left coset of the loop L by its proper subloop R too, i.e., for every $a, b \in L$, there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{1}$$

2.2 The concept of a transversal in a loop to its subloop. Elementary properties

Definition 5. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and let the left (right)* ***Condition A*** *be fulfilled. Let $\{R_x\}_{x\in E}$ be a set of all left (right) cosets in L to R that form a left (right) coset decomposition of the loop L. A set $T = \{t_x\}_{x\in E} \subset L$ is called a* ***left (right) transversal*** *in L to R if T is a complete set of representatives of the left (right) cosets R_x in L to R, i.e., there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$.*

Definition 6. *([6]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop, and both left and right* ***Conditions A*** *are fulfilled. A set $T = \{t_x\}_{x\in E}$ is called a* ***two-sided transversal*** *in L to R, if it is a left and a right transversal in L to R.*

Remark 7. *In Definitions 5, we assume that $t_1 = e$ (for simplicity and analogously to the case of transversals in groups).*

On a set E it is possible to define correctly the following operation:

$$x \overset{(T)}{\cdot} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = t_z \cdot r, \text{ where } t_x, t_y, t_z \in T, \quad r \in R, \quad (2)$$

if T is a left (non-reduced) transversal in L to R, and

$$x \overset{(T)}{\circ} y = z \quad \overset{def}{\Leftrightarrow} \quad t_x \cdot t_y = r \cdot t_z, \text{ where } \quad t_x, t_y, t_z \in T, \quad r \in R, \quad (3)$$

if T is a right (non-reduced) transversal in L to R.

Further we will study only the left transversals in L to R (the study of right transversals is similar).

Definition 8. *If a system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a loop, then the left transversal* $T = \{t_x\}_{x \in E}$ *is called a* ***loop transversal****.*

The following statement is known (see [6]):

Lemma 9. *A system* $< E, \overset{(T)}{\cdot}, 1 >$ *is a left loop with the two-sided unit 1.*

2.3 Representation of loop by cosets

Let $\langle L, \cdot, e \rangle$ be a loop, $\langle R, \cdot, e \rangle$ be its subloop, and let the left **Condition A** is fulfilled in $\langle L, \cdot, e \rangle$. Let $T = \{t_x\}_{x \in E}$ be a left transversal in L to R and let define the following mapping:

$$\begin{aligned} f \quad &: \quad L \times E \to E, \\ f \quad &: \quad (g, x) \to y = \hat{g}(x), \\ \hat{g}(x) \quad &= \quad y \quad \overset{def}{\Leftrightarrow} \quad g \cdot (t_x \cdot R) = t_y \cdot R. \end{aligned} \quad (4)$$

By virtue of the left **Condition A**, this definition (a *left action* of the loop L on a set E) is correct.

Lemma 10. *The mapping* $\hat{g}$ *is a permutation on the set* E*, for every element* $g \in L$*.*

It follows from the previous Lemma, that we obtained a permutation representation of a loop $\langle L, \cdot, e\rangle$. Indeed, we have the function $\varphi :$

$$\begin{aligned} \varphi &: \quad g \to \hat{g}, \\ \varphi &: \quad L \to \hat{L} \subset S_E, \end{aligned}$$

(where S_E is the symmetric group on a set E), and multiplication of permutations from $\hat{L}$ is defined as follows:

$$\hat{g}_1 * \hat{g}_2 = \hat{g}_3 \quad \overset{def}{\Leftrightarrow} \quad g_1 \cdot g_2 = g_3 \quad \text{in the loop } \langle L, \cdot, e\rangle. \tag{5}$$

Then it is easy to see that

$$\varphi(g_1) * \varphi(g_2) = \hat{g}_1 * \hat{g}_2 = \hat{g}_3 = \widehat{g_1 \cdot g_2} = \varphi(g_1 \cdot g_2),$$

i.e., φ is a homomorphism of the loop $\langle L, \cdot, e\rangle$ to the loop $\left\langle \hat{L}, *, \mathbf{id} \right\rangle$.

Lemma 11. *For an arbitrary left transversal $T = \{t_x\}_{x\in E}$ in a loop $L = \langle L, \cdot, e\rangle$ to its subloop $R = \langle R, \cdot, e\rangle$ the following statements are true:*

1. $\forall r \in R$*:* $\hat{r}(1) = 1$*;*
2. $\forall x, y \in E$*:*

$$\hat{t}_x(y) = x \overset{(T)}{\cdot} y, \quad \hat{t}_x^{-1}(y) = x\backslash y,$$

where $\hat{t}_x^{-1}$ is the inverse of the permutation $\hat{t}_x$ in S_E, and "$\backslash$" is the left division in the left loop $\langle E, \overset{(T)}{\cdot}, 1\rangle$. Moreover,

$$\hat{t}_x(1) = x, \quad \hat{t}_1(x) = x,$$
$$\hat{t}_x^{-1}(1) = x\backslash 1; \quad \hat{t}_x^{-1}(x) = 1.$$

Let us show how any two left transversals T and P in a loop L to its subloop R are connected.

Lemma 12 (see also [5]). *Let $T = \{t_x\}_{x \in E}$ and $P = \{p_x\}_{x \in E}$- be left transversals in L to R. Then there exists a set of elements $\{r_{(x)}\}_{x \in E}$ from R such that:*

1. $p_x = t_x r_{(x)}\ \forall x \in E$;

2. $x \overset{(P)}{\cdot} y = x \overset{(T)}{\cdot} \hat{r}^*_{(x)}(y)$, *where* $\hat{r}^*_{(x)}$ *is some suitable permutation from* $\hat{R}$.

This set $\{r_{(x)}\}_{x \in E}$ may be called a **derivation set** for transversal T (and for transversal operation $< E, \overset{(T)}{\cdot}, 1 >$).

Definition 13 (see [2]). *A groupoid $\langle E, \circ \rangle$ is called an isotope of a groupoid $\langle E, \cdot \rangle$ if there exists three permutations α, β, γ on E such that*

$$\gamma(x \cdot y) = \alpha(x) \circ \beta(y), \quad \forall x, y \in E.$$

In this case, the ordered triplet $\Phi = (\alpha, \beta, \gamma)$ is called an ***isotopy.*** *If $\gamma = id$, then the isotopy $\Phi = (\alpha, \beta, id)$ is called a* ***principal isotopy***.

Lemma 14. *If a loop $\langle E, \cdot, e_1 \rangle$ is isotopic to a loop $\langle E, \circ, e_2 \rangle$, then it is isomorfic to some principal isotop of a loop $\langle E, \circ \rangle$ (and this principal isotopy has the form $T_0 = (L_a^{-1}, R_b^{-1}, id), a \cdot b = e_2$).*

Remark 15. *If a loop $\langle E, \cdot, 1 \rangle$ is principal isotopic to a loop $\langle E, \circ, 1 \rangle$, then this principal isotopy has a form $T_0 = (R_b^{-1}, L_{a \diagdown 1}^{-1}, id)$ for some $a \in E$ ($^{-1}a = a \diagdown 1$ is the left inverse element to a in the loop $\langle E, \cdot, 1 \rangle$).*

3 The transformations which correspond to isotopy of the transversal operations

As follows from Lemma 14, to investigate loop transversals transformations, which correspond to isotopy of operations $\langle E, \overset{(T)}{\cdot}, 1 \rangle$ and

$\langle E, \overset{(P)}{\cdot}, 1\rangle$, it is enough to study the case of principal isotopy $T_0 = (L_a^{-1}, R_{a\diagdown 1}^{-1}, id)$ (as the transformations which correspond to isomorphism of transversal operations were studied earlier).

Let $T = \{t_x\}_{x\in E}$ and $P = \{p_x\}_{x\in E}$ be two loop transversals in a loop L to its subloop R, and $< E, \overset{(T)}{\cdot}, 1 >$ and $< E, \overset{(P)}{\cdot}, 1 >$ be transversal operations. Fix one of transversals, for example, $T = \{t_x\}_{x\in E}$.

Theorem 16. *Let the loops $\langle E, \overset{(T)}{\cdot}, 1\rangle$ and $\langle E, \overset{(P)}{\cdot}, 1\rangle$ be principal isotopic and this principal isotopy has the form $T_0 = (R_b^{-1}, L_a^{-1}, id)$ (note that $a \overset{(T)}{\cdot} b = 1$). Then*

$$\widehat{P} = \widehat{T} \cdot \widehat{t}_a^{-1}$$

for some $a \in E$.

Theorem 17. *Let $T = \{t_x\}_{x\in E}$ be a fixed loop transversal in L to R, and $a \in E$ be an arbitrary element from the set E. Define the following set of permutations:*

$$\widehat{p}_{x'} \overset{def}{=} \widehat{t}_x \widehat{t}_a^{-1} \quad \forall x \in E.$$

Then

1. *$P = \{p_{x'}\}_{x'\in E}$ is a left transversal in G to H;*

2. *The transversal operation $\langle E, \overset{(P)}{\cdot}, 1\rangle$ is principal isotopic to the operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$, and principal isotopy S has the following form: $S = (R_{a\diagdown 1}^{-1}, L_a^{-1}, id)$;*

3. *P is a loop transversal in L to R.*

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High Dimensional Inference – a Literature Review

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Abstract

High dimensional inference has been a crucial topic in statistical learning, especially with the rise of data that has a lot of features yet is difficult to gather.

We analyze regressions on data in large dimensions p that are much greater than the samples available, n. The usual method in constructing confidence intervals used for OLS is not useful when $p \gg n$ since asymptotic normality fails. We survey literature that comes up with new method of constructing these intervals.

Keywords: statistical learning, high-dimensional inference, confidence intervals, LASSO

1 Introduction

We will start discussing the case of high-dimensional linear regression with normal errors. Namely, we are given I.I.D. pairs $(Y_1, X_1), (X_2, Y_2), \dots (Y_n, X_n)$ which satisfy

$$Y_i = X_i^T \theta_0 + W_i \tag{1}$$

for $W_i \sim N(0, \sigma^2 I)$ and $\theta_0, Y_i, X_i \in \mathbb{R}^p$ for all $1 \leq i \leq n$. We denote $Y = (Y_1, Y_2, \dots, Y_n)$ and X to be the design matrix with rows $X_1^T, X_2^T, \dots X_n^T$. For simplicity, we denote $Z_j = (X_j, Y_j)$ and $Z = (X, Y)$. To study the high-dimensional case when $p \gg n$, LASSO regression has been introduced by [5] which promotes the sparsity of the learned coefficients via L1 regularization. Formally, the estimator of this regression is defined as

$$\hat{\theta}^n(X,Y;\lambda) = \arg\min_{\theta\in\mathbb{R}^p}\left\{\frac{1}{2n}\|Y-\theta X\|_2^2 + \lambda\|\theta\|_1\right\}. \quad (2)$$

Notably, this estimator comes with two drawbacks, especially in regards to building confidence intervals. First, the estimator is asymptotically biased, hence it is unclear how to center the confidence intervals. Second, it is difficult to find a distribution for it, hence building confidence intervals is theoretically complicated.

Another pitfall in high-dimensional regression is that the usual procedure of obtaining a reliable standard deviation estimator is no longer viable. Thankfully, this concern is addressed in [4] by providing the joint convex optimization problem,

$$(\hat{\theta}^n(\lambda),\hat{\sigma}(\lambda)) = \arg\min_{\theta\in\mathbb{R}^p,\sigma>0}\left\{\frac{1}{2\sigma n}\|Y-\theta X\|_2^2 + \frac{\sigma}{2} + \lambda\|\theta\|_1\right\}, \quad (3)$$

which produces $\hat{\sigma}(\lambda)$, a consistent estimator for σ when λ has suitable asymptotics that we will discuss later.

1.1 Understanding the bias

To address the first concern for $\hat{\theta}^n$ it is useful to study its relation to the true parameter, θ_0. As mentioned before, extracting a limiting distribution for $\hat{\theta}^n$ is not feasible, hence one has to look to concentration inequalities for a distance between $\hat{\theta}^n$ and θ_0. To do so, we first need to introduce some notation.

Definition 1. *We denote by $S \equiv supp(\theta_0)$ the support of θ_0, defined as $supp(\theta_0) = \{i \in [p] : \theta_{0,i} \neq 0\}$ and let $s_0 = |S|$.*

Definition 2. *For a symmetric matrix $\hat{\Sigma} \in \mathbb{S}_p$ and a set $S \in [p]$ we define the compatibility constant as*

$$\phi^2\left(\hat{\Sigma},S\right) \equiv \min_{\theta\in\mathbb{R}^p}\left\{\frac{|S|\theta^T\hat{\Sigma}\theta}{\|\theta_S\|_1^2} : \theta\in\mathbb{R}^p, \|\theta_{S^c}\|_1 \le 3\|\theta_S\|_1\right\},$$

and we further say that $\hat{\Sigma}$ satisfies the compatibility condition for a set $S \subseteq [p]$ with constant ϕ_0 if $\phi^2\left(\hat{\Sigma}, S\right) \geq \phi_0$. We say that it holds for design matrix X if it holds for $\hat{\Sigma} = \frac{1}{n}X^TX$.

We will also require a few technical conditions for the design matrix.

Assumption 1. *The design matrix X satisfies $C^{-1} \leq \lambda_{\min}(X) \leq \lambda_{\max}(X) \leq C$ for some constant, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the sample covariance matrix $\hat{\Sigma} = \frac{1}{n}X^TX$.*

Assumption 2. *The matrix $X\mathbb{E}[X^TX]^{-\frac{1}{2}}$ has independent sub-Gaussian rows with mean 0 and sub-Gaussian norm equal to some κ*

Assumption 3. *The maximum diagonal entry of $\hat{\Sigma}$ is at most K, formally $\max_i \frac{1}{n}(X^TX)_{ii} \leq K$.*

Definition 2 is rather technical, yet it enables us to introduce the following concentration inequality for an $\hat{\theta}^n$ inferred from a design matrix X that satisfies the compatibility condition for a set S with constant ϕ_0 and $|S| = S_0$ due to [1]

$$P\left(\|\hat{\theta}^n - \theta_0\|_1 \geq t^2 s_0\sqrt{\frac{\log(p)}{n}}\right) \leq 2p^{-C_0t^2/2}, \tag{4}$$

for some constant C_0 dependent on p, σ and K. While the imposed compatibility conditions on X seem restrictive, [2] proved in Theorem 7 that under Assumptions 1, 2 and 3 these conditions hold with probability at least $1 - e^{-C^3n}$ for some constant C_3 dependent on K, κ and C when $n \geq Ls_0\log(p/s_0)$ for a large enough constant L.

1.2 Debiasing the estimator

Since we can control the quantity $\|\hat{\theta}^n - \theta_0\|_1$ we want to build a better estimator for θ_0 via adding an asymptotically debiasing term. Formally, we want an $\hat{\theta}^u = \hat{\theta}^n + V$ for a term V that makes $\hat{\theta}^u - \theta_0$ to control.

Hence it makes sense to write $V = UX\hat{\theta}^n - UY$ since it helps us factorize as follows

$$\hat{\theta}^u - \theta_0 = \hat{\theta}^n - \theta_0 + UX(\hat{\theta}^n - \theta_0) - UW = (I + UX)(\hat{\theta}^n - \theta_0) - UW \tag{5}$$

Clearly, UW is an error term with mean 0. Hence, we want to minimize the quantity $\|I + UX\|_\infty$, which can attain 0 by setting $U = -\hat{\Sigma}^{-1}X^T$ if $\hat{\Sigma}$ is non-singular. However, this is not always the case when $p \gg n$, so [2] propose to set $U = -\frac{1}{n}MX^T$ where

$$M = \arg\min_{M \in \mathbb{R}^{p \times p}} \|M\hat{\Sigma} - I\|_\infty. \tag{6}$$

In this setting, M serves as a pseudo-inverse for $\hat{\Sigma}$ with respect to the infinity norm. [2] call $\mu_*(X) = \min_{M \in \mathbb{R}^{p \times p}} \|M\hat{\Sigma} - I\|_\infty$ the generalized coherence parameter of X and they prove in Lemma 23 that under Assumptions 1 and 2 we have the following concentration inequality

$$P\left(\mu_*(X) \geq a\sqrt{\frac{\log p}{n}}\right) \leq 2p^{-a^2 C_1}, \tag{7}$$

for a constant C_1 dependent on κ and C. Now, notice that due to Equations 4 and 7, under Assumptions 1, 2 and 3 we can deduce the following inequality on the residual term $\Delta := (M\hat{\Sigma} - I)(\theta_0 - \hat{\theta}^n)$ in 5:

$$P\left(\|\Delta\|_\infty \geq ac\frac{s_0 \log p}{n}\right) \leq 4e^{-C_3 n} + 4p^{-C_4 a^2 \wedge C_5 c^2}, \tag{8}$$

for some constants C_4 and C_5 dependent on κ and C. Notice that $4e^{-C_3 n}$ is the probability that X does not satisfies the compatibility conditions. This implies that $\|\Delta\|_\infty = O\left(\frac{s_0 \log p}{n}\right)$. Hence, as per the definition of U we have

$$\hat{\theta}^u = \hat{\theta}^n + \frac{1}{n}MX^T(Y - X^T\hat{\theta}^n), \tag{9}$$

and equation 5 can be rewritten as

$$\sqrt{n}(\hat{\theta}^u - \theta_0) = \sqrt{n}(M\hat{\Sigma} - I)(\theta_0 - \hat{\theta}^n) + \frac{1}{n}MX^TW, \tag{10}$$

where clearly $\frac{1}{n}MX^TW \sim N(0, \sigma^2 M\hat{\Sigma}M^T)$, so

$$\sqrt{n}(\hat{\theta}^u - \theta_0) - O\left(\frac{s_0 \log p}{\sqrt{n}}\right) \sim N(0, \sigma^2 M\hat{\Sigma}M^T), \tag{11}$$

which means that in the sparse regime $s_0 = o\left(\frac{\sqrt{n}}{\log p}\right)$ the estimator $\hat{\theta}^u$ introduced by [2] is indeed asymptotically unbiased.

1.3 Confidence intervals

We only need $\mu_*(X) = O\left(\sqrt{\frac{\log p}{n}}\right)$ to achieve the asymptotics in 11, hence in practice it is enough to impose that $\|\hat{\Sigma}m_i - e_i\|_\infty \leq a\sqrt{\frac{\log p}{n}} =: \mu$ for some parameter a. Recall that in 3 we introduced a consistent estimator for σ, so we can now build a $(1-\alpha)$ confidence interval for θ_0.

Algorithm 1 Computation of matrix the debiased coefficient - Algorithm 1 of [2]

Comment 1. **Input:** Design matrix X, parameters λ, μ and labels Y.
Initialize: $\hat{\Sigma} = \frac{1}{n}X^TX$ and $\hat{\theta}^n$ the solution of 2
for $i = 1$ **to** p **do**
  Let m_i be the solution of
  Minimize $m^T\hat{\Sigma}m$ subject to $\|\hat{\Sigma}m - e_i\|_\infty \leq 10\sqrt{\log p/n}$
end for
Set $M = (m_1, m_2, \dots m_p)^T$ if all the problems are feasible and I, the identity matrix, otherwise.
Output: $\hat{\theta}^u + \frac{1}{n}MX^T(Y - X^T\hat{\theta}^n)$

Theorem 1. *(Theorem 15 in [2]) Construct* $\hat{\theta}^u$ *as per Algorithm* 1*, with a large enough and let* $\hat{\sigma}$ *be the estimator of the standard deviation obtained from solving 3 with* $\lambda = 10\sqrt{2\log p/n}$*. Then, under Assumptions 1, 2, 3 and that* $s_0 = o\left(\frac{\sqrt{n}}{\log p}\right)$ *we have*

$$J_{i,\alpha}(Z,\theta_0) := \hat{\theta}^u \pm z_{1-\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}[M\hat{\Sigma}M^T]^{\frac{1}{2}}_{i,i} \tag{12}$$

satisfies $\lim_{n\to\infty} P(\theta_{0,i} \in J_{\alpha,i}(Z,\theta_0)) = 1-\alpha$.

As a closing remark for future readings, the authors of [2] discussed improved asymptotics towards their initial discoveries. In particular, they discuss the case when s_0 is of the order $o(n/(\log p)^2$ in the scenario when the covariance matrix is known apriori or can be efficiently estimated in their further work [3].

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Specific General Properties of Discrete W_q-Symmetry Groups

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Abstract

Groups of W_q-symmetry transformations of the bounded geometrical figures regularly weighted with "physical" tasks with orientation are studied in the present work. Some specific general properties of W_q-symmetry groups are briefly analyzed.

Keywords: groups, symmetry, quasi-homeomorphisms, crossed wreath products, $\overleftarrow{G}$-invariant subgroups.

1 Introduction

The generalized symmetry called W_q-symmetry was analyzed in [1], and the general theory of discrete groups of W_q-symmetry was elaborated and developed in [1-7]. In the case of W_q-symmetry, the geometrical component g of the transformation $g^{(w)} = wg$ operates both on points and on "indexes"-qualities, attributed to the points, just like in the case of $\overline{P}$-symmetry [8,9], while the supplementary substitution of "indexes" essentially depends on the choice of points just like in the case of W_p-symmetry [10,11]. In this work, some specific general features of groups of W_q-symmetry are analyzed.

2 On the description of the discrete groups of W_q-symmetry

Identifying the groups G and W with their isomorphic injections into $W \overset{\rightharpoonup}{\leftthreetimes} G = P \overset{\rightharpoonup}{\overline{\wr}} G$ by the rules $g \mapsto w_o g$, where w_o is the unit of the group W, and $w \mapsto w1$, where 1 is the unit of the group G, give us the posibility to find the subgroup $H = G^{(W_q)} \cap G$ of classical symmetry and the subgroup $V = G^{(W_q)} \cap W = G^{(W_q)} \cap W'$ of W-identical transformations from $G^{(W_q)}$.

Let $G^{(W_q)}$ be a group of W_q-symmetry with generating group G, initial transitive group of substitutions P, (i.e. $W = \overline{\prod}_{g_i \in G} P^{g_i}$, where $P^{g_i} \cong P$), subset $W' = \{w | g^{(w)} \in G^{(W_q)}\}$, the kernel H_1 of accompanying homomorphism $\tau : G \longrightarrow AutW$, symmetry subgroup H and the subgroup V of W-identical transformations.

The group $G^{(W_q)}$ is called major, minor or V-middle if $w_0 < V = W' = W, w_0 = V < W' = W$ or $w_0 < V < W' = W$, respectively. If W' is a non-trivial subgroup of W, then the group $G^{(W_q)}$ is called W'-semi-major, W'-semi-minor or (W', V)-semi-middle according to the cases when $w_0 < V = W', w_0 = V < W'$ or $w_0 < V < W'$. If $W' \subset W$, but W' is not a group, the group $G^{(W_q)}$ is called W'-pseudo-minor or (W', V)-pseudo-middle when $w_0 = V \subset W'$ or $w_0 < V \subset W'$ [1,8,11].

3 On specific properties of discrete W_q-symmetry groups

Any finite discrete group $G^{(W_q)}$ of W_q-symmetry satisfies the following specific properties:

1. The group operation in $G^{(W_q)}$ has the form $w_i g_i \otimes w_j g_j = w_k g_k$, where $g_k = g_i g_j$, $w_k = w_i^{g_j} \overset{\rightharpoonup}{\tau}_{g_i}(w_j), w_i^{g_j}(g_s) = w_i(g_j g_s)$ and $\overset{\rightharpoonup}{\tau}_{g_i}(w_j) = g_i w_j g_i^{-1}$;

2. The mapping λ of the group $G^{(W_q)}$ onto the group G by the rule $\lambda[g^{(w)}] = g$ is a homomorphism;

3. $W' = \{w | g^{(w)} \in G^{(W_q)}\}$ is a subset of group W; moreover, $w_0 \subset W' \subset W$, where w_0 is the unit of group W and W' in general is not a group.

4. The groups $G^{(W_q)}$ of W_q-symmetry are subgroups of the crossed standard Cartesian wreath product of P with the group G, accompanied by two morphisms: a) the isomorphism $\varphi : G \to AutW$ by the

rule $\varphi(g) = \overleftarrow{g}$, where automorphism $\overleftarrow{g}$ acts on the w elements, through left g-shift of their components ($\overleftarrow{g}: w \mapsto w^g$ and $w^g(g_k) = w(gg_k)$) and b) the homomorphism $\tau : G \to AutW$ by the rule $\tau(g) = \overrightarrow{\tau_g}$ (where $\overrightarrow{\tau_g}(w) = gwg^{-1}$);

5. The group $G^{(W_q)}$ contains as subgroup a group $G_1^{(W_1)}$ of $\overline{P}$-symmetry, where $W_1 = DiagW \bigcap W'$, from the family with the generating group G_1 ($G_1 < G$) and the same $Ker\tau = H_1$;

6. The group $G^{(W_q)}$ with the classical symmetry subgroup H contains as subgroup a group $H_1^{(W_p)}$ of W_p-symmetry from the family with the generating group H_1 and the symmetry subgroup H', where $H' = H \bigcap H_1$.

7. $V^g = V$ for any transformation $g^{(w)}$ from $G^{(W_q)}$, i.e. V is a $\overleftarrow{G}$-invariant subgroup;

8. $\overrightarrow{\tau_{g_i}}(V) = (w_i^{(g_j)})^{-1} V w_i^{g_j}$ for transformations $g_i^{(w_i)}$ and $g_j^{(w_j)}$ from $G^{(W_q)}$.

9. Any equivalent groups of W_q-symmetry have: a) the same generating group, b) the same initial group P of substitutions, c) conjugated symmetry subgroups, d) conjugated subgroups of W-identical transformations, e) equivalent subgroups of $\overline{P}$-symmetry, i) equivalent subgroups of W_p-symmetry.

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Combined Encryption Algorithm Using a Generalization of the Markovski Algorithm

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Abstract

This article is devoted to the construction of a generalization of a cryptographic algorithm based on the use of quasigroups of a special type. Here is a more detailed description of the algorithm proposed in [1]. An example is given to illustrate the operation of the developed algorithm. Our task was to show the effectiveness and advantages of using quasigroups of a special type in cryptology.

Keywords: T-quasigroup, encryption, decryption, Markovski algorithm, system of orthogonal operations, cryptosystem.

1 Introduction

This paper generalizes a cryptographic algorithm based on the use of quasigroups of a special form and provides an example illustrating the operation of this algorithm. This algorithm simultaneously uses two cryptographic procedures: encryption using a generalization of the Markovski algorithm [2] and encryption using a system of orthogonal operations. We present an implementation of this algorithm based on T-quasigroups, more precisely, based on medial quasigroups.

2 Basic concepts and definitions

The basic concepts, definitions, theorems and their consequences, as well as a detailed description of the algorithm and an example can be found in [3].

3 Combined encryption algorithm

Below we denote the action of the left (right, middle) translation to the power of a of a binary quasigroup (Q, g_1) on an element u_1 by the symbol ${}_{g_1}T^{a}_{l_1}(u_1)$. And so on. Here l_1 means the leader element. Note that the algorithm works for texts with even length.

Encryption. Initially, we have the plaintext $u_1, u_2, \ldots, u_{2n}$.

Step 1

$${}_{g_1}T^{a_{11}}_{l_1}(u_1) = v_1, {}_{g_2}T^{a_{12}}_{l_2}(u_2) = v_2, F^{a_{13}}_{1}(v_1, v_2) = (v_1^{'}, v_2^{'});$$

Step 2

$${}_{g_3}T^{a_{21}}_{v_1^{'}}(u_3) = v_3, {}_{g_4}T^{a_{22}}_{v_2^{'}}(u_4) = v_4, F^{a_{23}}_{2}(v_3, v_4) = (v_3^{'}, v_4^{'}); \cdots$$

Step N

$${}_{g_{2n-1}}T^{a_{n_1}}_{v_{2n-1}^{'}}(u_{2n-1}) = v_{2n-1}, {}_{g_{2n}}T^{a_{n_2}}_{v_{2n}^{'}}(u_{2n}) = v_{2n},$$

$$F^{a_{n_3}}_{n}(v_{2n-1}, v_{2n}) = (v_{2n-1}^{'}, v_{2n}^{'}).$$

We get the ciphertext $v_1^{'}, v_2^{'}, \ldots, v_{2n}^{'}$.

Thus, to encrypt a plaintext of length $2n$, we need $2n$ different operations on the quasigroups used in Combined encryption algorithm, n different values of the function F, 2 leader elements, and $3n$ different powers used in the translation algorithm.

Decryption. Let us have a ciphertext of the form $v_1^{'}, v_2^{'}, \ldots, v_{2n}^{'}$.

Step 1

$$F^{-a_{13}}_{1}(v_1^{'}, v_2^{'}) = (v_1, v_2), {}_{g_1}T^{-a_{11}}_{l_1}(v_1) = u_1, {}_{g_2}T^{-a_{12}}_{l_2}(v_2) = u_2;$$

Step 2

$$F^{-a_{23}}_{2}(v_3{'}, v_4{'}) = (v_3, v_4), {}_{g_3}T^{-a_{21}}_{v_1^{'}}(v_3) = u_3, {}_{g_4}T^{-a_{22}}_{v_2^{'}}(v_4) = u_4, \cdots$$

Step N

$$F_n^{-a_{n3}}(v'_{2n-1}, v'_{2n}) = (v_{2n-1}, v_{2n}),\ {}_{g_{2n-1}}T^{-a_{n1}}_{v'_{2n-1}}(v_{2n-1}) = u_{2n-1},$$

$${}_{g_{2n}}T^{-a_{n2}}_{v'_{2n}}(v_{2n}) = u_{2n}.$$

From Combined encryption algorithm we obtain the classical Markovski algorithm if we take only one quasigroup, one kind of quasigroup translation (left translations), each of which is taken to the first degree, and if the system of orthogonal operations (crypto procedures F) is not used. The Markovski algorithm is a special case of Combined encryption algorithm. The powers $a_{11}, a_{12}, \ldots, a_{n3}$ must be different to protect this algorithm from chosen plaintext and ciphertext attacks [4].

Additional information is required for the modifications proposed in this work [4, 5].

4 Example of Combined encryption algorithm

Consider the operation of Combined encryption algorithm based on T-quasigroups using a specific example.

Example *Take a cyclic group $(Z_{1237}, +) = (A, +)$. Initially, we have a plaintext $u_1, u_2, u_3, u_4, u_5, u_6$ of length $k = 6$. First, we define all the T-quasigroups we need:*

$$1.(A, g_1) = (A, *) : x * y = 513 \cdot x + 293 \cdot y + 115;$$

$$(A, *) \perp (A, \overset{(13)}{*}) : x \overset{(13)}{*} y = 938 \cdot x + 1017 \cdot y + 986.$$

$$2.(A, g_2) = (A, \circ) : x \circ y = 161 \cdot x + 1007 \cdot y + 509;$$

$$(A, \circ) \perp (A, \overset{(23)}{\circ}) : x \overset{(23)}{\circ} y = 1114 \cdot x + 1038 \cdot y + 1094.$$

$$3.(A, g_3) = (A, \star) : x \star y = 715 \cdot x + 201 \cdot y + 698;$$

$$(A, \star) \perp (A, \overset{(23)}{\star}) : x \overset{(23)}{\star} y = 298 \cdot x + 1157 \cdot y + 175.$$

$$4.(A, g_4) = (A, \Diamond) : x \Diamond y = 1111 \cdot x + 751 \cdot y + 263;$$

$$(A, \Diamond) \perp (A, \overset{(13)}{\Diamond}) : x \overset{(23)}{\Diamond} y = 1129 \cdot x + 703 \cdot y + 1190.$$

$$5.(A, g_5) = (A, \odot) : x \odot y = 451 \cdot x + 3 \cdot y + 511;$$

$$(A, \odot) \perp (A, \overset{(13)}{\odot}) : x \overset{(23)}{\odot} y = 96 \cdot x + 949 \cdot y + 424.$$

$$6.(A, g_6) = (A, \circledast) : x \circledast y = 941 \cdot x + 992 \cdot y + 189;$$

$$(A, \circledast) \perp (A, \overset{(23)}{\circledast}) : x \overset{(23)}{\circledast} y = 948 \cdot x + 722 \cdot y + 849.$$

We define 3 systems of two parastrophic orthogonal T-quasigroups:

$$1.F_1(x, y) = (123 \cdot x + 719 \cdot y + 839, 1125 \cdot x + 1077 \cdot y + 644),$$

$$F_1^{-1}(x, y) = (1210 \cdot x + 876 \cdot y + 315, 390 \cdot x + 848 \cdot y).$$

$$2.F_2(x, y) = (401 \cdot x + 711 \cdot y + 911, 671 \cdot x + 896 \cdot y + 164),$$

$$F_2^{-1}(x, y) = (858 \cdot x + 951 \cdot y + 44, 172 \cdot x + 1066 \cdot y).$$

$$3.F_3(x, y) = (1001 \cdot x + 19 \cdot y + 299, 152 \cdot x + 823 \cdot y + 321),$$

$$F_3^{-1}(x, y) = (858 \cdot x + 380 \cdot y, 566 \cdot x + 1009 \cdot y + 440).$$

Now we can use all the operations described above in Combined encryption algorithm.

Let us choose the elements as leaders: $l_1 = 599; l_2 = 1001$.

In the algorithm, we use the following values of the degrees: $a_{11} = 3; a_{12} = 1; a_{13} = 1; a_{21} = 2; a_{22} = 1; a_{23} = 2; a_{31} = 3; a_{32} = 1; a_{33} = 2$.

Encryption.

Let the plaintext look like: **101; 213; 405; 79; 615; 923.**

Step 1

$$v_1 = {}_{g_1}T^3_{599}(101) = 418, v_2 = {}_{g_2}T^1_{1001}(213) = 113,$$

$$(v_1', v_2') = F_1^2(418, 113) = (1141, 72),$$

Step 2

$$v_3 = {}_{g_3}T^2_{1141}(405) = 837, v_4 = {}_{g_4}T^1_{72}(79) = 1086,$$

$$(v_3^{'}, v_4^{'}) = F_2^2(837, 1086) = (511, 96),$$

Step 3

$$v_5 = {}_{g_5}T^3_{511}(615) = 1179, v_6 = {}_{g_6}T^1_{96}(923) = 460,$$

$$(v_5^{'}, v_6^{'}) = F_3^2(1179, 460) = (866, 751).$$

We get the following ciphertext: 1141; 72; 511; 96; 866; 751.

The decryption of the text is checked in a similar way.

A more detailed description of the algorithm and an example of encryption and decryption of text can be found in [4].

5 Conclusion

A program has been developed that uses a free version of the Pascal ABC programming language. The conducted experiments show that encoding/decoding is performed quite quickly. The program works for any values of leaders, powers, and any plaintext of length 6. It can be easily modified for text of any length and any used quasigroups and functions.

Combined encryption algorithm makes it possible to obtain an almost "natural" stream cipher, i.e., a stream cipher that encodes a pair of plaintext elements at any step. It is easy to see that Combined encryption algorithm can be generalized to the n-ary case.

Proper binary groupoids are preferable to linear quasigroups in terms of the construction of the map $F(x, y)$ for greater encryption security, but in this case, decryption may be slower than in the case of linear quasigroups, and the definition of these groupoids requires more memory. The same remark is valid for the choice of the function g. Perhaps the golden mean in this choice problem is the use of linear quasigroups over non-Abelian, especially simple, groups.

In this cipher, there is a possibility of protection against the standard statistical attack. For this area, more commonly used letters or pairs of letters can be denoted by more than one integer or more than one pair of integers [4].

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Combinatorial Symmetry of Twisty Puzzles and Twist Axes of Order 4, 6, 8, 9, 10 and 11

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Abstract

This paper introduces the notion of conbinatorial symmetry of twisty puzzles, presents geometric principles of searching for twisty puzzles with sun mechanism. Additionally, some geometrical and combinatorial properties of such puzzles and their polyhedra are discussed.

Keywords: combinatorial symmetry, twisty puzzles, sun mechanism.

1 Introduction

Ernő Rubik created the puzzle known today as Rubik's Cube to teach his students about algebraic groups. Today, there are many puzzles similar to Rubik's Cube, which are generally named *twisty puzzles*. Such puzzles have usually symmetric form. However, the group of permutations plays much more important role in twisty puzzles. Still the moves of a twisty puzzle have clear geometric interpretation and thus have much in common with puzzle symmetry.

There are two important differences between symmetry and combinatorics of twisty puzzles. First, their moves are twists of one part relative to another rather than rotations. And unlike rotations, the composition of twists is usually not a twist. So, the permutation group of twisty puzzles is richer than their symmetry group. Second, for physical realization of twisty puzzles the exact geometry of parts is not necessary. The puzzles with approximate geometry are named *fudged*.

If the difference between the exact geometric quantity, say angle, and physically possible is of the order of magnitude of parts tolerance, the puzzle can be constructed.

To emphasize the similarity with symmetry, consider *combinatorial symmetry.*

Definition. *Combinatorial symmetry of a twisty puzzle is the group of moves (the orbit) of its individual parts including all different ways a part can be placed.*

The order of combinatorial symmetry group of each part is usually as large as the order of even symmetry of puzzle polyhedron. However, the combinatorial symmetry can be greatly enriched by introducing partial twists. These twists change the puzzle shape, such puzzles are named *jumbling.* One way to achieve this is to allow half-twists of puzzle faces, such puzzles are named *sun* twisty puzzles. Sun mechanism doubles the order of twist axes.

2 Main results

The article [1] presents two principles that help finding all possible sun twisty puzzles:

Result 1: *The polyhedra suitable for sun twisty puzzles are dual to those which have all faces regular (not necessarily equal) and whose all vertex defects are equal.* They include regular solids, some Catalan solids, bipyramids, trapezohedra and certain Johnson solids duals. Some examples of puzzles with shape from the last category are Cullinan (truncated pentagonal bipyramid), Big Dipper (trapezo-rhombic dodecahedron) and Cygnus (pseudo-deltoidal icositetrahedron).

Such polyhedra have all symmetric vertices. All described until now sun puzzles are based only on polyhedra with 3-fold and 4-fold vertices. Sun puzzles have clusters of four 3-fold vertex pieces in place of 4-fold vertices. If polyhedra with 5-fold vertices can be considered for sun puzzles, they would form 5-fold faces, so their 5-fold vertices would be truncated. Puzzles with only standalone 3-fold vertices have exact geometry, with clusters of 4 vertices have fudged geometry.

Result 2: *When the puzzle is based on polyhedron with only 3-fold and 4-fold vertices, its number of faces f depends only on twist axes order n as:*

$$f = \frac{24}{12 - n}. \tag{1}$$

Despite the fact sun puzzles were initially introduced by doubling the axes order, there exist also sun puzzles with odd axes order. Table 1 presents integer solutions of (1).

Table 1. Face number f of puzzle as function on axes order n

n	f	Complexity c	Examples and comments
0	2	–	No obvious geometrical realization
4	3	1/2	Spherical trigonal prismoid
6	4	1	Tetrahedron, tetragonal prism
8	6	2	Cube, trigonal bipyramid
9	8	3	Octahedron, tetragonal trapezohedron
10	12	5	Dodecahedron, rhombododecahedron, truncated pentagonal bipyramid, trapezo-rhombic dodecahedron
11	24	11	Pentagonal icositetrahedron, deltoidal icositetrahedron, pseudo-deltoidal icositetrahedron
12	∞	–	2D puzzles

3 Conclusion and further work

The attempt to enumerate all sun twisty puzzles together with their geometrical and combinatorial properties are approached in [2].

The inner cut of vertex pieces forms always almost right angle. In the cluster of four vertices, the inner edges of adjacent parts are almost parallel and form almost regular pattern. Are these coincidence or there are geometrical properties that ensure these aspects?

If we divide all sun puzzles in categories by axes order then inside each group, the following numbers seem to be invariant:

Table 2. Apparent invariants of each group of sun puzzles

Invariant	Description and comments
n	Axes order
f	Number of faces
$v + 2e$	Number of vertex parts plus double of number of edges
$v_4 + e$	Number of 4-fold vertices plus number of edges

The last two invariants from Table 2 above are also related with complexity from Table 1 as:

$$c = \frac{v + 2e}{16} = \frac{v_4 + e}{6}. \quad (2)$$

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Some Types of LCA Groups with Local Ring of Continuous Endomorphisms

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Abstract

We describe the structure of some locally compact abelian (LCA) groups whose ring of continuous endomorphisms is local.

Keywords: LCA groups, rings of continuous endomorphisms of LCA groups, local rings, topologically indecomposable groups, purely topologically indecomposable groups, co-purely topologically indecomposable groups, LCA groups with finite exchange property.

1 Introduction

Let $\mathcal{L}$ be the class of LCA groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of all continuous endomorphisms of X. One may ask:

For which groups $X \in \mathcal{L}$, the ring $E(X)$ is local,

i.e., its non-invertible elements form an ideal.

A. Orsatti in [1], P.A. Krylov in [2], and R.B. Warfield in [3] have characterized various types of discrete groups in $\mathcal{L}$ with this property.

In the following, we will answer the corresponding question for different non-discrete types of groups in $\mathcal{L}$.

2 The results

First we introduce some notations. If $X \in \mathcal{L}$, we let $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and X^* denote respectively the connected component of

X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the torsion subgroup of X, and the character group of X with its usual topology. We will use the group of rationals $\mathbb{Q}$ taken discrete, and the group of reals $\mathbb{R}$ with its usual topology. If $p \in \mathbb{P}$ and $n \in \mathbb{N}$, we denote by $\mathbb{Z}(p^n)$ the cyclic group of order p^n, by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group, and by $\mathbb{J}_p$ the group of p-adic integers, all taken discrete. Further, we denote by $\mathbb{Z}_p$ the group of p-adic integers with its unique compact topology, and by $\mathbb{Q}_p$ the group of all p-adic numbers with its usual locally compact topology.

Theorem 1. *Let $X \in \mathcal{L}$. If $E(X)$ is local, then either X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$, or else X is residual.*

Corollary 1. *Let X be a non-residual group in $\mathcal{L}$. The following conditions are equivalent:*

(i) *$E(X)$ is a field.*

(ii) *$E(X)$ is local.*

(iii) *X is topologically isomorphic with one of the groups $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$.*

Definition 1.

(i) *The pseudo-socle of an abelian group X, denoted by $soc(X)$, is the pure subgroup of X generated by all minimal pure fully invariant subgroups of X.*

(ii) *A group $X \in \mathcal{L}$ is said to be purely topologically indecomposable if every closed, pure subgroup of X is topologically indecomposable.*

Theorem 2. *Let X be a residual, purely topologically indecomposable group in $\mathcal{L}$ with the following properties:*

1) *$t(X)$ is closed in X.*

2) *If $k(X) \neq \{0\}$ and $d(X) = \{0\}$, then $t(X) \neq \{0\}$.*

The following conditions are equivalent:

(i) $E(X)$ *is a discrete valuation domain.*

(ii) $E(X)$ *is local.*

(iii) X *is topologically isomorphic either with a pure subgroup* G *of* $\mathbb{J}_p$ *such that* $soc(G) = G$*, or with one of the groups* $\mathbb{Q}_p$*,* $\mathbb{Z}(p^\infty)$*, or* $\mathbb{Z}(p^n)$*, where* $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$*.*

Definition 2.

(i) *The pseudo-radical,* $rad(X)$*, of a connected group* $X \in \mathcal{L}$ *is the connected component of the intersection of all maximal connected topologically fully invariant closed subgroups of* X*. If* X *has no maximal connected topologically fully invariant closed subgroups, then* $rad(X)$ *is taken to be all of* X*.*

(ii) *A group* $X \in \mathcal{L}$ *is said to be co-purely topologically indecomposable if for any closed, pure subgroup* C *of* X*, the quotient group* X/C *is topologically indecomposable.*

Corollary 2. *Let* X *be a residual, co-purely topologically indecomposable group in* $\mathcal{L}$ *with the following properties:*

1) $\overline{rX}/\bigcap_{n\in\mathbb{N}_0}\overline{nX}$ *is compact for some* $r \in \mathbb{N}_0$*.*

2) *If* $c(X) \neq X$ *and* $m(X) = X$*, then* $\bigcap_{n\in\mathbb{N}_0}\overline{nX} \neq X$*.*

The following conditions are equivalent:

(i) $E(X)$ *is a discrete valuation domain.*

(ii) $E(X)$ *is local.*

(iii) X *is topologically isomorphic either with a quotient group of* $\mathbb{J}_p^*$ *by a closed pure subgroup* A *with* $rad(A) = \{0\}$*, or with one of the groups* $\mathbb{Q}_p$*,* $\mathbb{Z}_p$*, or* $\mathbb{Z}(p^n)$*, where* $p \in \mathbb{P}$ *and* $n \in \mathbb{N}$*.*

Definition 3. *Let $\mathcal{S}$ be a subclass of $\mathcal{L}$ closed under taking finitary topological direct products, closed subgroups, and topological isomorphisms. A group X in $\mathcal{S}$ is said to have the finite exchange property in $\mathcal{S}$ if X satisfies the following condition:*

For any group Y in $\mathcal{S}$ and any its pair of topological direct sum decompositions

$$Y = X' \oplus G = Y_1 \oplus \cdots \oplus Y_n,$$

where $G, Y_1, \ldots, Y_n \in \mathcal{S}$, $n \in \mathbb{N}_+$, and $X' \cong X$, there exist closed subgroups Y_i' and Z_i of Y_i such that $Y_i = Y_i' \oplus Z_i$ for all $i \in \{1, \ldots, n\}$, and $Y = X' \oplus Y_1' \oplus \cdots \oplus Y_n'$.

Theorem 3. *Let X be a non-zero topologically indecomposable group in $\mathcal{L}$. The ring $E(X)$ is local if and only if X has the finite exchange property in $\mathcal{L}$.*

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About Quadratic Functional Equations on Quasigroups

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Abstract

Functional equations over binary quasigroups are under consideration. An equation is called: *quadratic*, if each individual variable has either two appearances or none; *cancellable*, if a variable has two appearances and another none in a proper subterm; *reducible*, if it is equivalent to a system of equations such that every of which has less number of individual variables than the given one. Only the functional equations of unipotency, commutativity, associativity and mediality are noncancellable, and the irreducible ones are the same except the mediality. The criteria for the parastrophic equivalency of the equations up to the noncancellable equations were found.

Keywords: quasigroup, quadratic functional equation, reducible, cancelable, parastrophically equivalent equations.

1 Introduction

The concept of functional equation is taken from J.Aczél [1]. This article studies a class of functional equations over a quasigroup environment (briefly, qen), i.e. over a set $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2$ of functions defined on the same carrier Q, where Φ_0 is a set of nullary operations, Φ_1 a group of bijections, Φ_2 is a set of binary quasigroup operations and the set Φ is closed under compositions of binary and unary or nullary operations as well as unary and nullary operations from Φ. Moreover, we restrict our attention only to quadratic functional equations (each individual variable appears twice or never).

The considered problem is to find conditions when a functional equation can be reduced to some shorter functional equations. This problem was under consideration in many works [2], [3], [4], [5] [6], [7]. We continue the research in this area, but we propose another approach for solving this problem.

2 Preliminaries

A functional equation is called: 1) *reducible*, if it is equivalent to a system of equations every of which has less number of individual variables than the given one; 2) *cancelable*, if it has a subterm having all appearances of an individual variable and none of another; 3) *parastrophically cancelable*, if it is parastrophically equivalent to a cancelable functional equation.

3 Equations in 1 and 2 individual variables

The equations $F(x,x) = a$ and $F(x,y) = F(y,x)$ are said to be a functional equation of *idempotency* and commutativity, respectively.

Theorem 1. *Quadratic equations in* 1 *individual variable are parastrophically noncancellable and parastrophically equivalent to the functional equation of idempotency.*

Theorem 2. *Let* $\omega = \upsilon$ *be a quadratic quasigroup functional equation in* 2 *individual variables, then the following propositions are equivalent:*
1) the equation $\omega = \upsilon$ *is parastrophically noncancellable;*
2) a subterm of the length 2 is repetirion-free;
3) the equation $\omega = \upsilon$ *is parastrophically equivalent to commutativity.*

4 Equations in 3 and 4 individual variables

The functional equations

$$F_1(F_2(x,y),z) = F_3(x,F_4(y,z)),$$

$$F_1(F_2(x,y),F_3(z,v)) = F_4(F_5(x,z),F_6(y,v)).$$

are said to be a equation of *associativity* and *mediality*, respectively.

Theorem 3. *Let $\omega = v$ be a quadratic quasigroup functional equation in 3 individual variables, then the following propositions are equivalent:*
1) the equation $\omega = v$ is parastrophically noncancellable;
2) every subterm (including outer one) of the length 2,3 is repetirion-free and two subterms of the length 2 have exactly 1 variable in common;
3) the equation $\omega = v$ is parastrophically equivalent to the functional equation of associativity.

Theorem 4. *Let $\omega = v$ be a quadratic quasigroup functional equation in 4 individual variables, then the following statements are equivalent:*
1) the equation $\omega = v$ is parastrophically noncancellable;
2) the equation satifies all of the following conditions:
2.1) its subterms of the length 2, 3 and 4 (including the outer ones) are repetition-free;
2.2) any 2 subterms of the length 2 has different sets of individual variables;
2.3) if $xy \cdot z$ is a subterm, then neither xz, nor yz is a subterm;
2.4) if $(xy \cdot z)u$ is a subterm, then zu is not a subterm;
2.5) if xy is a subterm, then there is no u such that xu and yu are subterms.
3) the equation $\omega = v$ is parastrophically equivalent to the functional equation of mediality.

5 Conclusion

Each noncancellable quadratic functional equation in n individual variables is parastrophically equivalent to unipotency if $n = 1$; commutativity if $n = 2$; assiciativity if $n = 3$; mediality if $n = 4$. It does not exist if $n > 4$. Any functional equation of mediality is reducible.

The atoms of the semilattice of quadratic varieties are varieties being defined by the parametrical unipotency, commutativity and associativity.

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About Multiple Prolongations of Latin Cubes

Fedir Sokhatsky, Daria Moroz

Abstract

We propose an algorithm of prolongation of a Latin cube with k new elements ($k \geq 1$).

Key words: Latin cube, prolongation of Latin cube, two-dimensional transversal, one-dimensional transversal, ternary quasigroup, prolongation of ternary quasigroup, multiple prolongation of Latin cube.

1 Introduction

Consider a cube in the coordinate system $Oxyz$. Each cell is denoted by (x, y, z, u), where (x, y, z) are coordinates of the cell and u is the element (the *forth coordinate*) located in this cell. So, the Latin cube C is denoted by

$$C := \{(x, y, z, u) \mid 0 \leqslant x, y, z, u < m\}. \tag{1}$$

The row plane, column plane, and string plane of the cube defined by an element a is denoted by the equations $x = a$, $y = a$, $z = a$ respectively. And the row, column and string is denoted by a pair of the planes, i.e., by their intersection:

$$y = a, z = b; \qquad x = a, z = b; \qquad x = a, y = b.$$

If any cell of a set $T \subset C$ is uniquely defined by: 1) an arbitrary pair of its coordinates, then T is called a *two-dimensional transversal*; 2) an arbitrary coordinate, then T is called the *one-dimensional transversal* of this Latin cube. The two-dimensional transversal of a Latin cube is a Latin square, and an arbitrary transversal of the square is the one-dimensional transversal of the Latin cube.

2 Prolongation algorithm on k elements

The purpose of this algorithm is to prolong a Latin cube C of order m on k new elements that are denoted by u_0, ..., u_{k-1}. Let the cube has k two-dimensional transversals τ_i and so each of them has k one-dimensional transversals $\theta_{ij} \subset \tau_i$, where i, $j = 0$, ..., $k-1$.

Step 1. Add $3k$ new planes with empty cells to C:

$$x = m + i, \quad y = m + i, \quad z = m + i, \qquad i = 0, ..., k-1.$$

Step 2. We transfer all elements of two-dimensional transversals, except the elements of the one-dimensional transversals, to the corresponding added planes. Namely, let $i \in \{0, \ldots, k-1\}$ and the cell (a, b, c, d) belong to the two-dimensional transversal τ_i, but it does not belong to all of the one-dimensional transversals $\theta_{i,j}$, $j = 0, \ldots, k-1$. We transfer element d to cells $(m+i, b, c)$, $(a, m+i, c)$, $(a, b, m+i)$ and we put the new element u_i into the cell (a, b, c).

Step 3. Let the cell (a, b, c, d) belong to a one-dimensional transversal $\theta_{i,j}$. We copy the element d into the cells of the added planes and so d belongs to the following cells:

$$\begin{array}{cc} (a, b, c, d), & (a, m+i, m+j, d), \\ (m+i, b, m+j, d), & (m+i, m+j, c, d). \end{array}$$

We fill the cells $(m+i, b, c)$, $(a, m+i, c)$, $(a, b, m+i)$ with the new element u_j.

Step 4. $\{(m+i, m+j, m+s) \mid i, j, s = 0, \ldots, k-1\}$ is an arbitrary Latin cube of the order k filled by the elements u_0, ..., u_{k-1}.

Step 5. The prolongation is over.

3 Example

We prolong the initial cube C on the set $\{0, 1, 2, 3, 4\}$ to the final cube D on the set $\{0, 1, 2, 3, 4, 5\}$ (Fig. 1).

The two-dimensional transversal τ (pink and red circles) and the one-dimensional transversal θ (red circles) are selected. The new element is denoted by 5 (light blue circles).

Step 1. Add three new planes $x = 5$, $y = 5$, $z = 5$ to the cube C.

Step 2. We transfer all elements of two-dimensional transversals τ, except the elements of the one-dimensional transversals θ, to the corresponding added planes. Namely, let $(a, b, c, d) \in \tau$, but $(a, b, c, d) \notin \theta$. We transfer element d to cells $(5, b, c)$, $(a, 5, c)$, $(a, b, 5)$ and we put the new element 5 into the cell (a, b, c):

$$(5, b, c, d), \quad (a, 5, c, d), \quad (a, b, 5, d), \quad (a, b, c, 5) \in D.$$

Step 3. Let $(a, b, c, d) \in \theta$. We copy the element d into the cells of the added planes:

$$(a, b, c, d), \quad (a, 5, 5, d), \quad (5, b, 5, d), \quad (5, 5, c, d) \in D.$$

The cells $(5, b, c)$, $(a, 5, c)$, $(a, b, 5)$ is filled with a new element 5:

$$(5, b, c, 5), \quad (a, 5, c, 5), \quad (a, b, 5, 5) \in D.$$

Step 4. Put a new element 5 in cell $(5, 5, 5)$: $(5, 5, 5, 5) \in D$.

Step 5. Prolongation is over. The cube D of order 6 has been formed.

4 Conclusion

In this work for the first time, the algorithm for prolonging Latin cubes of order m on k new elements has been found and it is illustrated when $m = 5$ and $k = 1$.

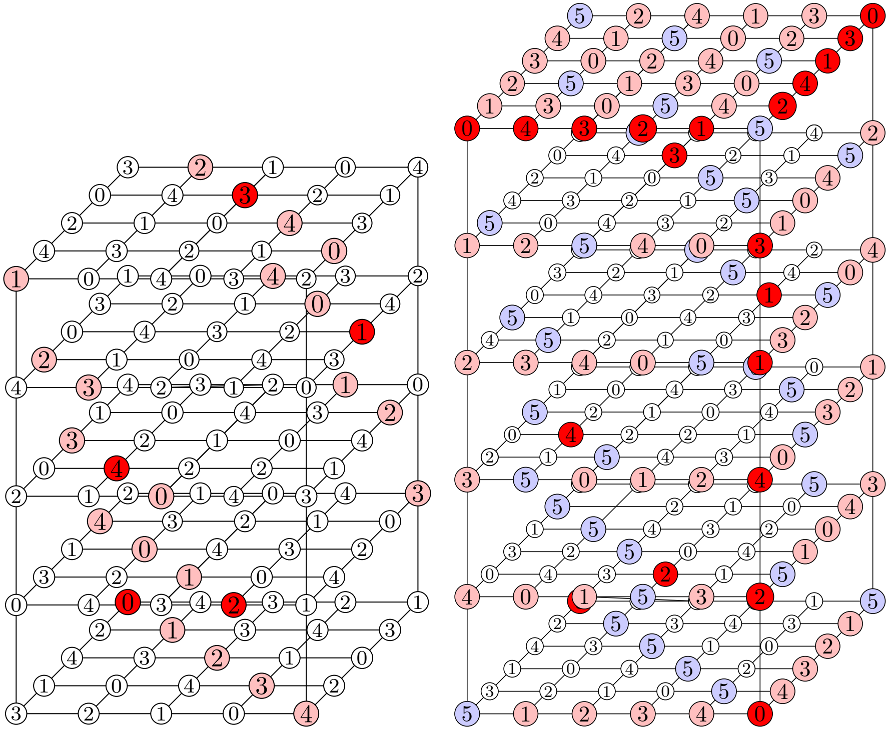

Figure 1. A prolongation of the cube C of order 5 to the cube D of the order 6

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On Self-Orthogonal n-ary Quasigroups

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Abstract

We consider self-orthogonal finite n-ary quasigroups and give some estimations of their spectrum in the present work. A method of construction of self-orthogonal n-quasigroups, using self-orthogonal quasigroups of lower arity, is given. In particular, it is shown that there exist: 1) self-orthogonal 2^k-ary quasigroups of every order $q > 3, q \neq 6$, where $k \geq 1$; 2) self-orthogonal p^k-quasigroups of prime order p, for every $p \geq 3$ and every $k \geq 1$; 3) self-orthogonal $2n$-quasigroups of order q, for every $q > 3, q \neq 6$ and $n + 1 \not\equiv 0 \pmod{q}$.

Keywords: orthogonal n-operations, parastrophic-orthogonal n-quasigroups, self-orthogonal n-quasigroups.

The Euler conjecture about non-existence of orthogonal latin squares of order $n \equiv 2 \pmod 4$ solved, negatively for $n \geq 10$ of this type, in the middle of 20th century, led to new concepts, types of orthogonality and new problems concerning orthogonality and parastrophic-orthogonality of operations (quasigroups). A system of n-ary operations $A_1, A_2, ..., A_n$, defined on a set Q, is called orthogonal if the set of equations $\{A_i(x_1, x_2, ..., x_n) = a_i\}_{i=\overline{1,n}}$ has a unique solution for every $a_1, a_2, ..., a_n \in Q$. A system of s n-ary operations defined on a set Q, where $s \geq n$, is orthogonal if every n its operations are orthogonal. The orthogonality of n-ary operations (quasigroups, latin hypercubes) was considered from algebraic and/ or combinatorial point of view by many authors [1].

An n-ary groupoid (Q, A) is an n-ary quasigroup if in the equality $A(x_1, x_2, \ldots, x_n) = x_{n+1}$ every element of $\{x_1, x_2, \ldots, x_{n+1}\}$ is uniquely determined by the remaining n. The operation denoted by

${}^{\sigma}\!A$, where $\sigma \in S_{n+1}$ and A is an n-ary quasigroup operation, defined by the equivalence

$${}^{\sigma}\!A(x_{\sigma 1}, x_{\sigma 2}, ..., x_{\sigma n}) = x_{\sigma(n+1)} \Leftrightarrow A(x_1, x_2, ..., x_n) = x_{n+1},$$

is called a parastrophe of A. If $\sigma(n+1) = n+1$, than ${}^{\sigma}\!A$ is called a principal parastrophe. An n-ary quasigroup is parastrophic-orthogonal (self-orthogonal) if there exist n orthogonal its parastrophes (principal parastrophes). It is known that there exist self-orthogonal latin squares of any order $q \neq 1, 2, 3, 6$ [2]. The complete characterization of the spectrum of parastrophic-orthogonal, in particular self-orthogonal quasigroups, in the n-ary case is still opened [3-7]. We consider new constructions of self-orthogonal finite n-ary quasigroups and give some estimations of their spectrum.

Let $A_1, A_2, ..., A_n$ be n-ary operations ($n \geq 2$), defined on a set Q. It follows from the definition that the n-ary operations $A_1, A_2, ..., A_n$ are orthogonal if and only if the mapping $\theta : Q^n \mapsto Q^n, \theta(x_1^n) = (A_1(x_1^n), A_2(x_1^n), ..., A_n(x_1^n))$, is a bijection, where by (x_1^n) we denoted the sequence $(x_1, x_2, ..., x_n)$. If Q is a finite set then the operations $A_1, A_2, ..., A_n$ are orthogonal if and only if the solution of the system of equations $\{A_i(x_1, x_2, ..., x_n) = a_i\}_{i=\overline{1,n}}$ is unique for each $a_1, a_2, ..., a_n \in Q$ (which is equivalent to the injectivity of θ).

Proposition 1. *If $(Q,\ A)$ is a finite self-orthogonal n-quasigroup and ${}^{\alpha_1}\!A,\ {}^{\alpha_2}\!A,\ ...,\ {}^{\alpha_n}\!A$ is an orthogonal system of its principal parastrophes then the n^2-ary groupoid $(Q,\ B_1)$, where $B_1(x_1^{n^2}) ={}^{\alpha_1}\!A({}^{\alpha_1}\!A(x_1^n),$ ${}^{\alpha_1}\!A(x_{n+1}^{2n}), ...,\ {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2}))$ is a self-orthogonal quasigroup.*

Proof. Let $(Q,\ A)$ be a finite self-orthogonal n-quasigroup and let ${}^{\alpha_1}\!A,\ {}^{\alpha_2}\!A,\ ...,\ {}^{\alpha_n}\!A$ be an orthogonal system of its principal parastrophes. We consider the n^2-ary operations $B_1, B_2, \ldots, B_{n^2}$ defined below:

$$\begin{cases} B_1(x_1^{n^2}) &=& {}^{\alpha_1}\!A({}^{\alpha_1}\!A(x_1^n),\ {}^{\alpha_1}\!A(x_{n+1}^{2n}), ...,\ {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2})), \\ ... & ... & ... \\ B_n(x_1^{n^2}) &=& {}^{\alpha_n}\!A({}^{\alpha_1}\!A(x_1^n),\ {}^{\alpha_1}\!A(x_{n+1}^{2n}), ...,\ {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2})), \\ B_{n+1}(x_1^{n^2}) &=& {}^{\alpha_1}\!A({}^{\alpha_2}\!A(x_1^n),\ {}^{\alpha_2}\!A(x_{n+1}^{2n}), ...,\ {}^{\alpha_2}\!A(x_{n(n-1)+1}^{n^2})), \\ ... & ... & ... \\ B_{n^2}(x_1^{n^2}) &=& {}^{\alpha_n}\!A({}^{\alpha_n}\!A(x_1^n),\ {}^{\alpha_n}\!A(x_{n+1}^{2n}), ...,\ {}^{\alpha_n}\!A(x_{n(n-1)+1}^{n^2})). \end{cases}$$

The operations B_i, $\overline{1,\, n}$ are principal parastrophes of the quasigroup $(Q,\, B_1)$. Indeed, as ${}^{\alpha_1}\!A,\ {}^{\alpha_2}\!A,\ ...,\ {}^{\alpha_n}\!A$ are principal parastrophes of A, we get that each of the operations $B_1,\, B_2,\, ...,\, B_{n^2}$ is a principal parastrophe of the quasigroup operation

$$D(x_1^{n^2}) = A(A(x_1^n),\, A(x_{n+1}^{2n}, ...,\, A(x_{n(n-1)+1}^{n^2}).$$

Using the transitivity of the relation of parastrophy, we obtain that each of the operations B_i, $i = \overline{1,\, n^2}$ is a principal parastrophe of $(Q,\, B_1)$.

As Q is a finite set, the system $\{B_1,\, B_2,\, ...,\, B_{n^2}\}$ is orthogonal if the solution of the system of equations

$$B_i(x_1^{n^2}) = a_i,\ i = \overline{1,\, n^2}.$$

is unique. Let $\{B_i(x_1^{n^2}) = B_i(y_1^{n^2})\}$, $i = \overline{1,\, n^2}$, then

$$\begin{cases} {}^{\alpha_1}\!A({}^{\alpha_1}\!A(x_1^n),\, ...,\, {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2})) & = & {}^{\alpha_1}\!A({}^{\alpha_1}\!A(y_1^n),\, ...,\, {}^{\alpha_1}\!A(y_{n(n-1)+1}^{n^2})), \\ \dots & \dots & \dots \\ {}^{\alpha_n}\!A({}^{\alpha_1}\!A(x_1^n),\, ...,\, {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2})) & = & {}^{\alpha_n}\!A({}^{\alpha_1}\!A(y_1^n),\, ...,\, {}^{\alpha_1}\!A(y_{n(n-1)+1}^{n^2})), \\ \dots & \dots & \dots \\ {}^{\alpha_1}\!A({}^{\alpha_n}\!A(x_1^n),\, ...,\, {}^{\alpha_n}\!A(x_{n(n-1)+1}^{n^2})) & = & {}^{\alpha_1}\!A({}^{\alpha_n}\!A(y_1^n),\, ...,\, {}^{\alpha_n}\!A(y_{n(n-1)+1}^{n^2})), \\ \dots & \dots & \dots \\ {}^{\alpha_n}\!A({}^{\alpha_n}\!A(x_1^n),\, ...,\, {}^{\alpha_n}\!A(x_{n(n-1)+1}^{n^2})) & = & {}^{\alpha_n}\!A({}^{\alpha_n}\!A(y_1^n),\, ...,\, {}^{\alpha_n}\!A(y_{n(n-1)+1}^{n^2})). \end{cases}$$

As the system of parastrophes $\{{}^{\alpha_1}\!A,\, ...,\, {}^{\alpha_n}\!A\}$ is orthogonal we obtain

$$\begin{cases} {}^{\alpha_1}\!A(x_1^n) & = & {}^{\alpha_1}\!A(y_1^n), \\ \dots & \dots & \dots \\ {}^{\alpha_n}\!A(x_1^n) & = & {}^{\alpha_n}\!A(y_1^n), \\ \dots & \dots & \dots \\ {}^{\alpha_1}\!A(x_{n(n-1)+1}^{n^2}) & = & {}^{\alpha_1}\!A(y_{n(n-1)+1}^{n^2}), \\ \dots & \dots & \dots \\ {}^{\alpha_n}\!A(x_{n(n-1)+1}^{n^2}) & = & {}^{\alpha_n}\!A(y_{n(n-1)+1}^{n^2}) \end{cases} \Rightarrow \begin{cases} x_1 & = & y_1, \\ \dots & \dots & \dots \\ x_n & = & y_n, \\ \dots & \dots & \dots \\ x_{n(n-1)+1} & = & y_{n(n-1)+1}, \\ \dots & \dots & \dots \\ x_{n^2} & = & y_{n^2} \end{cases}$$

hence $\{B_1,\, B_2,\, ...,\, B_{n^2}\}$ is orthogonal, i.e. $(Q,\, B_1)$ is a n^2-ary self-orthogonal quasigroup. □

Example 1. The ternary quasigroup $(\mathbb{Z}_p,\, A)$, where $A(x_1,\, x_2,\, x_3) = x_1 + x_2 - x_3$ and p is a prime number, is self-orthogonal and A, ${}^{(13)}\!A$, ${}^{(23)}\!A$ is an orthogonal system of its principal parastrophes. According to Proposition 1, the 9-ary quasigroup $(\mathbb{Z}_p, B_1)$, where $B_1 = A(A(x_1^3), A(x_4^6), A(x_7^9))$ is self-orthogonal. An orthogonal system of principal parastrophes of $(\mathbb{Z}_p, B_1)$ is given by $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9$, where

$$\begin{array}{l}
B_2(x_1^9) = {}^{(13)}A(A(x_1^3), A(x_4^6), A(x_7^9)) = {}^{(17)(28)(39)}B_1(x_1^9);\\
B_3(x_1^9) = {}^{(23)}A(A(x_1^3), A(x_4^6), A(x_7^9)) = {}^{(47)(58)(69)}B_1(x_1^9);\\
B_4(x_1^9) = A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(13)(46)(79)}B_1(x_1^9);\\
B_5(x_1^9) = {}^{(13)}A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(19)(28)(37)(46)}B_1(x_1^9);\\
B_6(x_1^9) = {}^{(23)}A({}^{(13)}A(x_1^3), {}^{(13)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(13)(49)(58)(67)}B_1(x_1^9);\\
B_7(x_1^9) = A({}^{(23)}A(x_2^3), {}^{(23)}A(x_4^6), {}^{(13)}A(x_7^9)) = {}^{(23)(56)(89)}B_1(x_1^9);\\
B_8(x_1^9) = {}^{(13)}A({}^{(23)}A(x_1^3), {}^{(23)}A(x_4^6), {}^{(23)}A(x_7^9)) = {}^{(17)(29)(38)(56)}B_1(x_1^9);\\
B_9(x_1^9) = {}^{(23)}A({}^{(23)}A(x_1^3), {}^{(23)}A(x_4^6), {}^{(23)}A(x_7^9)) = {}^{(23)(47)(68)}B_1(x_1^9).
\end{array}$$

Proposition 2. *Let (Q, A) and (Q, B) be finite n-ary, and respectively m-ary, self-orthogonal quasigroups. If $\{{}^{\alpha_1}A, {}^{\alpha_2}A, ..., {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B, {}^{\beta_2}B, ..., {}^{\beta_m}B\}$ are orthogonal systems of principal parastrophes then (Q, C_1), where*

$$C_1(x_1^{mn}) = {}^{\beta_1}B({}^{\alpha_1}A(x_1^n), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn}))$$

is a self-orthogonal nm-ary quasigroup.

Proof. Let (Q, A) and (Q, B) be finite n-ary, and respectively m-ary, self-orthogonal quasigroups and let $\{{}^{\alpha_1}A, {}^{\alpha_2}A, ..., {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B, {}^{\beta_2}B, ..., {}^{\beta_m}B\}$ be orthogonal systems of their principal parastrophes. We consider the following nm-ary operations:

$$\left\{\begin{array}{rcl}
C_1(x_1^{mn}) & = & {}^{\beta_1}B({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})),\\
... & ... & ...\\
C_m(x_1^{mn}) & = & {}^{\beta_m}B({}^{\alpha_1}A(x_1^n), {}^{\alpha_1}A(x_{n+1}^{2n}), ..., {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})),\\
... & ... & ...\\
C_{n(m-1)+1}(x_1^{mn}) & = & {}^{\beta_1}B({}^{\alpha_n}A(x_1^n), {}^{\alpha_n}A(x_{n+1}^{2n}), ..., {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})),\\
... & ... & ...\\
C_{mn}(x_1^{mn}) & = & {}^{\beta_m}B({}^{\alpha_n}A(x_1^n), {}^{\alpha_n}A(x_{n+1}^{2n}), ..., {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})).
\end{array}\right.$$

Remark that the system $\{C_1, C_2, ..., C_{mn}\}$ is orthogonal. Indeed, as Q is a finite set, it is sufficient to show that, for each $c_1, c_2, ..., c_{nm} \in Q$, the solution of the system of equations

$$C_i(x_1^{mn}) = c_i,\ i = \overline{1, mn}$$

is unique. Let $\{C_i(x_1^{mn}) = C_i(y_1^{mn})\}$, $i = \overline{1,\, mn}$, then

$$\begin{cases} {}^{\beta_1}B({}^{\alpha_1}A(x_1^n),\, ...,\, {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})) & = & {}^{\beta_1}B({}^{\alpha_1}A(y_1^n),\, ...,\, {}^{\alpha_1}A(y_{n(m-1)+1}^{mn})), \\ ... & ... & ... \\ {}^{\beta_n}B({}^{\alpha_1}A(x_1^n),\, ...,\, {}^{\alpha_1}A(x_{n(m-1)+1}^{mn})) & = & {}^{\beta_n}B({}^{\alpha_1}A(y_1^n),\, ...,\, {}^{\alpha_1}A(y_{n(m-1)+1}^{mn})), \\ ... & ... & ... \\ {}^{\beta_1}B({}^{\alpha_n}A(x_1^n),\, ...,\, {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})) & = & {}^{\beta_1}B({}^{\alpha_n}A(y_1^n),\, ...,\, {}^{\alpha_n}A(y_{n(m-1)+1}^{mn})), \\ ... & ... & ... \\ {}^{\beta_n}B({}^{\alpha_n}A(x_1^n),\, ...,\, {}^{\alpha_n}A(x_{n(m-1)+1}^{mn})) & = & {}^{\beta_n}B({}^{\alpha_n}A(y_1^n),\, ...,\, {}^{\alpha_n}A(y_{n(m-1)+1}^{mn})). \end{cases}$$

As the system of principal parastrophes $\{{}^{\beta_1}B,\, ...,\, {}^{\beta_m}B\}$, is orthogonal, we get that

$$\begin{cases} {}^{\alpha_1}A(x_1^n) & = & {}^{\alpha_1}A(y_1^n), \\ ... & ... & ... \\ {}^{\alpha_n}A(x_1^n) & = & {}^{\alpha_n}A(y_1^n), \\ ... & ... & ... \\ {}^{\alpha_1}A(x_{n(m-1)+1}^{mn}) & = & {}^{\alpha_1}A(y_{n(m-1)+1}^{mn}), \\ ... & ... & ... \\ {}^{\alpha_n}A(x_{n(m-1)+1}^{mn}) & = & {}^{\alpha_n}A(y_{n(m-1)+1}^{mn}). \end{cases}$$

Now, using the fact that the system $\{{}^{\alpha_1}A,\, ...,\, {}^{\alpha_n}A\}$, is orthogonal, we obtain $x_i = y_i$, $i = \overline{1,\, mn}$, i.e. the system $\{C_1,\, C_2,\, ...,\, C_{mn}\}$ is orthogonal.

As $\{{}^{\alpha_1}A,\, ...,\, {}^{\alpha_n}A\}$ and $\{{}^{\beta_1}B,\, ...,\, {}^{\beta_m}B\}$ are principal parastrophes of A, and respectively B, we have that each of the operations $\{C_1,\, C_2,\, ...,\, C_{mn}\}$ is a principal parastrophe of D, where $D(x_1^{mn}) = B\left(A(x_1^n),\, ...,\, A(x_{n(m-1)+1}^{mn})\right)$, so each of them is a principal parastrophe C_1, i.e. $(Q,\, C_1)$ is a self-orthogonal mn-ary quasigroup. $\square$

Corollary 1. *If there exist n-ary self-orthogonal quasigroups of order q then there exist n^k-ary self-orthogonal quasigroups of order q, for every $k \geq 2$.*

Corollary 2. *There exist 2^k-ary self-orthogonal quasigroups of any order $q \neq 1, 2,\, 3,\, 6\}$, for every $k \geq 1$.*

Corollary 2 follows from Proposition 2 and the fact that there exist self-orthogonal latin squares of any order $q \neq 1, 2, 3, 6$. [2]

Example 2. *The set of all principal parastrofes of the n-ary quasigroup $(\mathbb{Z}_q,\, A)$, where $A(x_1^n) = 2x_1 + x_2 + ... + x_n$,* is $A, {}^{(12)}A, ..., {}^{(1n)}A$. Moreover, this system of principal parastrophes is orthogonal if and only if $n+1 \not\equiv 0 \pmod{q}$. Hence, there exist self-orthogonal n-ary quasigroups of order $q \geq 3$ for any $n \geq 2$ such that $n+1 \not\equiv 0 \pmod{q}$.

Corollary 3. *There exist self-orthogonal p^k-ary quasigroups of order p for every odd prime p and every $k \geq 1$.*

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Some Applications of Properties of the Extensions of the Ring of Integers

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Abstract

This article presents some considerations related to the application of ring algebra concepts to solving problems proposed in mathematics competitions. The content can be useful to mathematics teachers and high school students.

Keywords: Gaussian integers, division theorem, factorization, greatest common divisor.

1 Introduction

The topics we will study are extensions of common properties of the Gaussian integers $Z[i] = \{a + bi | a, b \in Z\}$.

The objective of this investigation aims at the didactic transposition of the mathematical contents for the familiarization of students with some applications of concepts related to extension of the ring Z. Some properties of this structure are successfully used in solving problems. For this purpose, the following facts can be examined: the norm on $Z[i]$; divisibility in $Z[i]$; the division theorem in $Z[i]$; the Euclidean algorithm in $Z[i]$; unique factorization in $Z[i]$; primes in $Z[i]$; modular arithmetic in $Z[i]$; applications of $Z[i]$ to the arithmetic of Z.

Definition 1. *For $\alpha = a + bi \in Z[i]$, its norm is the product $N(\alpha) = (a + bi)(a - bi) = a^2 + b^2$.*

The norm of every Gaussian integer is a non-negative integer, but it is not true that every non-negative integer is a norm. Indeed, the norms are the integers of the form $a^2 + b^2$ and not every positive integer is a sum of two squares. It's preferable to deal with norms on $Z[i]$ instead of absolute values on $Z[i]$ because norms are integer.

Definition 2. *We say β divides $\alpha \in Z[i]$ (and write $\beta|\alpha$) if $\alpha = \beta\gamma$ for some $\gamma \in Z[i]$.*

Definition 3. *For non-zero α and $\beta \in Z[i]$ a greatest common divisor of α and β (denoted $gcd(\alpha, \beta)$) is a common divisor with maximal norm.*

We can speak about **a greatest** common divisor, but not about **the greatest** common divisor. If δ is a greatest common divisor of α and β, then its unit multiples $-\delta, i\delta, -i\delta$ (at least) are greatest divisors too.

Definition 4. *When α and $\beta \in Z[i]$ only have unit factors in common, we call them relatively prime.*

Some contest problems can be solved using the Eisenstein's ring of integers $Z[\omega] = \{\alpha = a + b\omega | a, b \in Z\}$, where $\omega = \dfrac{-1 + i\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$. Its norm is the product $N(\alpha) = a^2 - ab + b^2$. Units in this ring are $\pm 1, \pm\omega, \pm(1 + \omega) = \mp\omega^2$.

2 Some properties of the concepts in $Z[i]$

Theorem 1. *The norm is multiplicative: for α and β in $Z[i]$, $N(\alpha\beta) = N(\alpha)N(\beta)$.*

Theorem 2. *If $N(x)$ is prime, then x is also prime.*

Corollary 1. *The only Gaussian integers which are invertible in $Z[i]$ are $\{-i; i; -1; 1\}$.*

Theorem 3. *A Gaussian integer for $\alpha = a + bi$ is divisible by an ordinary integer c if and only if $c|a$ and $c|b$ in Z.*

We may prove that Euclidean Algorithm holds in this rings. Not every unique factorization domain (UFD) is Euclidean, but every Euclidean ring is a UFD. We won't prove either of these facts, some of the proofs can be found in [1]. Lets move onto problem examples.

3 Applications of algebraic number fields to Olympiad problem solving

We will start with a problem proposed for IMO. For these problems we will need the following important lemma.

Lemma. If a, b, c, d are elements of a unique factorization domain K, satisfying $ab = cd$, then there exist $m, n, p, q \in K$, such that $a = mn$, $b = pq$, $c = mp$, $d = nq$.

Proof. Let $gcd(a, c) = n$, then since $a \mid cd$, $\frac{a}{n} = m \mid d$, analogously if $gcd(b, c) = p$, then $\frac{b}{p} = q \mid d$. We get $a = mn$, $b = pq$, $c = mp$, $d = nq$. Notice since we are working in UFD all operations are well defined.

More notes on application of this lemma may be found in [2].

Problem 1. Let x, y and z be natural numbers satisfying $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$ and $z = ac + bd$.

Solution. We will factor the right side of our equation and apply previously stated lemma to get $x = uv$, $y = tw$, $z + i = uw$, $z - i = vt$, for some $u, v, t, w \in Z[i]$. Because x and y are natural numbers, we must have $v = q\overline{u}$ and $t = p\overline{w}$ with $p, q \in Q$. Since $z + i = uw$ and $z - i = vt$, we conclude that $p = q = 1$. Now setting $u = a + bi$, $w = c - di$ yields $x = a^2 + b^2$, $y = c^2 + d^2$ and $z = ac + bd$.

We would like to continue with the classic Fermat's Last Theorem for the case $n = 3$. Proof presented below is due to Gauss.

Problem 2. Prove that the equation $x^3 + y^3 = z^3$ has no solutions in integers.

Solution. Suppose that x, y, z are nonzero elements of $Z[\omega]$ (the ring of Eisenstein's integers, where $\omega = \frac{-1 + i\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$), which satisfy the equation. We can assume that they are pairwise coprime. Lets consider $\delta = 1 - \omega$. It is prime and $\overline{\delta} = 1 - \omega^2 = (1 + \omega)\delta$. For some $\phi \in Z[\omega]$, $\delta \mid \phi$ iff $\delta \mid \overline{\phi}$. We also want to prove that every number in $Z[\omega]$ is congruent to $-1, 0$ or $1 \pmod{\delta}$. It follows from the congruence $a + b\omega \equiv (a + b) \mod \delta$ and $a + b = 3q + r \equiv r$

(mod δ), where $r \in \{-1, 0, 1\}$. Another important fact that we will use is that $\phi \equiv \pm 1 \pmod{\delta}$ (with $\phi \in Z[\omega]$) implies $\phi^3 \equiv \pm 1 \pmod{\delta^4}$. Returning back to our initial equation we notice that exactly one of the numbers x, y, z must be divisible by δ, because otherwise we get that x, y, z are congruent to $\pm 1 \pmod{\delta^4}$, which implies false congruence. We assume without loss of generality that $\delta \mid z$. Also $\delta^4 \mid z^3$, so $\delta^2 \mid z$.

Let $k \geq 2$ be the smallest natural number for which there exists solution with x, y, z pairwise coprime and $\delta^k \mid z$, but δ^{k+1} does not divide z. Factoring the equation we obtain $z^3 = (x+y)(\omega x+\omega^2 y)(\omega^2 x+\omega y)$, noticing that the factors are congruent $\pmod{\delta}$ and their sum is 0, we may denote them as $x+y = A\delta$, $\omega x+\omega^2 y = B\delta$ and $\omega^2 x+\omega y = C\delta$, where $A, B, C \in Z[\omega]$ are pairwise coprime and $A+B+C = 0$. Because $ABC = \left(\frac{z}{\delta}\right)^3$ is a perfect cube we must have $A = \varepsilon\alpha^3$, $A = \zeta\beta^3$, $A = \eta\gamma^3$, for some pairwise coprime $\alpha, \beta, \gamma \in Z[\omega]$ and $\varepsilon, \zeta, \eta \in \{\pm 1, \pm\omega, \pm\omega^2\}$. Therefore $\varepsilon\alpha^3 + \zeta\beta^3 + \eta\gamma^3 = 0$. Since $\varepsilon\zeta\eta$ is a unit and a perfect cube it must be equal to ± 1. On the other hand, $ABC = \left(\frac{z}{\delta}\right)^3$ is divisible by δ (because $\delta^2 \mid z$), so exactly one of the numbers α, β, γ is divisible by δ, let it be γ. In fact, $\gamma^3 \mid ABC$ and $\delta^{3k-3} \mid ABC$, and δ^{3k-2} does not divide ABC. So, $\delta^{k-1} \mid \gamma$ and δ^k does not divide γ. Now since α, β are not divisible by δ, evaluating $\varepsilon\alpha^3 + \zeta\beta^3 + \eta\gamma^3 = 0 \pmod{\delta^4}$ gives us $\varepsilon \pm \zeta \equiv 0 \pmod{\delta^4}$, therefore $\varepsilon = \pm\zeta$ and $\varepsilon\zeta\eta = \pm 1$, implies $\eta = \pm\zeta$. Canceling ζ in equation yields $\alpha^3 \pm \beta^3 \pm \gamma^3 = 0$, which is another nontrivial solution to the initial equation with α, β, γ being pairwise coprime. However, in this solution $\delta^{k-1} \mid \gamma$ and δ^k does not divide γ, which contradicts choice of k.

Problem 3. Prove that a prime number p can't be written as a sum of 2 squares of positive integers in more than one way.

Solution. Let p be a prime number, which contradicts our assumption. Then we have $p = x^2 + y^2 = u^2 + v^2$, factoring yields $(x + iy)(x - iy) = (u + iv)(u - iv)$. Denoting $a = x + iy$, $b = u + iv$ the former condition may be rewritten as $a\overline{a} = b\overline{b}$. If $gcd(a, b) = n$, then $m = \frac{a}{n}$ must divide $\overline{b}$, following this logic we get $a = nm =$

$(\alpha+i\beta)(\gamma+i\delta)$ and $b = n\overline{m} = (\alpha+i\beta)(\gamma-i\delta)$. Now $x = Re(a) = \alpha\gamma-\beta\delta$ and $y = Im(a) = \beta\gamma + \alpha\delta$, going back to our first relation we get $p = x^2+y^2 = (\alpha\gamma-\beta\delta)^2+(\beta\gamma+\alpha\delta)^2 = (\alpha^2+\beta^2)(\gamma^2+\delta^2)$. For p to be prime one of the factors must be equal to 1, without loss of generality lets assume $\alpha^2+\beta^2 = 1$, this implies that $\alpha = 0, \beta = 1$ or $\alpha = 1, \beta = 0$. If $\alpha = 0$ then $x = -\delta$ and $u = \delta$, contradiction with assumption that integers are positive. If $\beta = 0$ the $x = \gamma = u$ and obviously $y = v$, contradiction with assumption that integers are different. So there aren't any primes p representable as sum of two squares in two different ways.

Problem 4. *(IMO 2001, Problem 6)* Let $a > b > c > d$ be positive integers such that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

Solution. Initial condition is equivalent to $a^2-ac+c^2 = b^2+bd+d^2$, which may be factored as $(a + \omega c)(a + \omega^2 c) = (b - \omega d)(b - \omega^2 d) \Leftrightarrow (a + \omega c)\overline{(a + \omega c)} = (b - \omega d)\overline{(b - \omega d)}$. Lets denote $a + \omega c = x$ and $b - \omega d = y$. If $gcd(x, y) = n$ and $\frac{x}{n} = m$, then $m \mid \overline{y}$, resulting $gcd(x, \overline{y}) = m$. So we get $x = nm$, $y = n\overline{m}$ and $n = p+\omega q$, $m = r+\omega s$. Now going backwards we find that $a = pr - qs$, $c = qr + ps - qs$, $b = pr + qs - ps$ and $d = ps - qr$. Now $ab + cd = (pr - qs)(pr + qs - ps) + (qr + ps - qs)(ps - qr) = (p^2 - q^2)(r^2 - rs + s^2)$. For $ab+cd$ to be prime we must have $p^2 - q^2 = \pm 1$ or $r^2 - rs + s^2 = \pm 1$. First equation has solutions only if one of numbers p, q is 0, which would imply that $a + b = 0$ or $c = d$. Second equation has solutions only if one of the numbers r, s is 0 or $r = s$, all three cases leading to contradictions to initial condition. So $ab + cd$ is product of two integers with absolute values greater than 1, i.e. is composite.

Problem 5. *(BMO Shortlist 2010)* Find all pairs of integers (x, y), such that $x^3 = 2y^2 + 1$.

Solution. Rewrite original equation as $x^3 = (1+y\sqrt{-2})(1-y\sqrt{-2})$. Since $Z[\sqrt{-2}]$ is UFD and $gcd(1 + y\sqrt{-2}, 1 - y\sqrt{-2}) = 1$. We get that $1+y\sqrt{-2} = (u+v\sqrt{-2})^3 = u(u^2-6v^2)+v(3u^2-2v^2)\sqrt{-2}$. Therefore, $u(u^2 - 6v^2) = 1$, which implies $u = 1$ and $v = 0$. Consequently, $y = 0$ and $x = 1$, which are obviously solutions to the given equation.

4 Conclusion

This approach to solving some Olympiad problems is interesting because involves the application of algebraic facts, not studied in high school, but accessible to students for understanding. Through the didactic transposition of mathematical contents, high school students get familiar with various possibilities for making generalizations and abstractions. The contents give them the opportunity to reflect on the nature of mathematical knowledge.

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List of Normal 3-Isohedral Spherical Tilings for Group Series $2 * n$

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Abstract

Tilings of the 2-dimensional sphere with disks which fall into 3 transitivity classes under the group action are studied. For isometry group series $2 * n$, $n = 1, 2, \ldots$, we list all normal by Grünbaum and Shephard 3-isohedral spherical tilings.

Keywords: 3-isohedral tiling, sphere, normal tiling, isometry group series, Delone class.

We begin with reminding basic concepts. Let W be a tilng of the 2-dimensional sphere with topological disks and G be a discrete isometry group of the sphere.

Definition 1. *The tiling W is called k-isohedral with respect to the group G if G maps W onto itself and the tiles of W fall into exactly k transitivity classes under the group G.*

Definition 2. *Consider all possible pairs (W, G) where the tiling W of the sphere is k-isohedral with respect to the group G. Two pairs (W, G) and (W', G') are said to belong to the same Delone class if there exists a homeomorphism φ of the sphere which maps the tiling W onto the tiling W' and for isometry groups the relation $G = \varphi^{-1}G'\varphi$ holds.*

In a tiling, a connected component of the intersection of two and more different disks is defined to be a vertex of the tiling if it is a single point or to be an edge of the tiling otherwise. A Delone class is fundamental if the group G acts simply transitively on the set of tiles or is non-fundamental otherwise.

For spherical groups we use the Conway's orbifold symbol, characterizing the orbifold which is the quotient of the sphere by the group.

There are 7 countable series and 7 sporadic discrete isometry groups of the sphere.

A tiling of the sphere is called normal by B. Grünbaum and G. C. Shephard [1] if it satisfies the following conditions:

SN1. Each tile is a topological disk.

SN2. The intersection of any set of tiles is a connected (possibly empty) set.

SN3. Each edge of the tiling has two endpoints which are vertices of the tiling.

Both fundamental and non-fundamental isohedral (1-isohedral) tilings of the sphere with disks are described in [1].

In [2] the author developed some general methods for obtaining k-isohedral ($k \geq 2$) tilings of the Euclidean plane, the sphere and the hyperbolic plane. In particular, the splitting procedure can be applied to fundamental $(k-1)$-isohedral tilngs with disks yielding fundamental k-isohedral tilings with disks. By this way the enumeration of Delone classes of fundamental 2-isohedral tilings of the sphere with disks was obtained [3]. Also the same methods in terms of Delaney–Dress symbols were implemented in algorithms and computer programs [4, 5].

Recently the author has applied the splitting procedure to all 29 series of Delone classes of fundamental 2-isohedral tilings of the sphere with disks for group series $2 * n$, $n = 1, 2, \ldots$. It has resulted in 499 series of Delone classes of fundamental 3-isohedral spherical tilings with disks. The number 499 coincides with the numerical results given in [5].

We select from the obtained 3-isohedral tilings those which satisfy the above normality conditions SN1–SN3. Below we list the obtained Delone classes of 3-isohedral tilings by indicating, for each Delone class, 3 cycles of vertex valencies of tiles.

For group series $2 * n$, $n = 1, 2, \ldots$, that corresponds to series of 3-dimensional point isometry groups $\widetilde{2N} \cdot m = \widetilde{2N} : 2$, $N = 1, 2, \ldots$, there are 64 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3^4; 3^3.4^2; 3^2.4.2n.4]$, $[3^4; 3^2.4^2; 3^3.4.2n.4]$, $[3^4; 3^2.4.4n; 3^3.4.4n]$, $[3^2.4^2; 3.4.2n.4; 3^3.4^2]$, $[3^3.6; 3^2.4.6; 3.4.2n.6]$, $[3.4^3; 3.4^3; 3.4.2n.4]$,

$[3^3; 3^2.4.2n.4; 3^4.4^2]$, $[3^2.4; 3^2.4.2n.4; 3.4.3.4^2]$, $[3.4^2; 3^3.4^2; 3^2.4.2n.4]$, $[3^3; 3^2.4^2; 3^4.4.2n.4]$, $[3^3; 3^2.4.4n; 3^4.4.4n]$, $[3^2.4; 3^2.4^2; 3.4.3.4.2n.4]$, $[3.4^2; 3^2 4^2; 3^3.4.2n.4]$, $[3^2.4; 3^2.4.4n; 3.4.3.4.4n]$, $[3^2.6; 3.4.2n.6; 3^3.4.6]$, $[3.4^2; 3^2.4.4n; 3^3.4.4n]$, 2 different series $[3.4^2; 3.4^3; 3.4^2.2n.4]$, $[3.4^2; 3.4.2n.4; 3.4^4]$, $[3.4.6; 3^3.6; 3^2.4.2n.6]$, $[3.4.4n; 3^2.4^2; 3^3.4.4n]$, $[3.4.4n; 3^2.4.4n; 3^3.4^2]$, $[4^2.2n; 3^2.4^2; 3.4^4]$, $[3.4^2; 3.4^2.4n; 3.4^2.4n]$, $[3.4^2; 3.4.6^2; 3.4.2n.6]$, 2 different series $[3.4^2; 3.4.4n.4; 3.4.4n.4]$, $[3.4.6; 3.4.6.4; 3.4.2n.6]$, $[3.4.6; 3.4.6^2; 3.4.2n.4]$, $[3.4.4n; 3.4^3; 3.4^2.4n]$, $[3.4.6n; 3^3.6n; 3^2.4.6n]$, $[3.6.4n; 3^3.6; 3^2.6.4n]$, $[4^3; 4^4; 4^3.2n]$, $[4^2.2n; 4^4; 4^4]$, $[4.6.2n; 3^2.4.6; 3.4^2.6]$, $[3^2.6; 3.4.6; 3^3.4.2n.6]$, $[3.4^2; 4^2.2n; 3^2.4^2]$, $[3^2.6; 3.6.4n; 3^3.6.4n]$, $[3^2.6n; 3.4.6n; 3^3.4.6n]$, $[3.4.6; 3.4.6; 3.4.6.2n.4]$, $[3.4.6; 4.6.2n; 3^2.4^2.6]$, $[3.4.4n; 3.4.4n; 3.4^4]$, $[4^3; 4^3; 4^4.2n]$, $[4^3; 4^2.2n; 4^5]$, $[3.4.6; 3.6.4n; 3.4.6.4n]$, $[3.4.8; 3.8^2; 3.4.2n.8]$, $[3.4.4n; 3.6.4n; 3.4.6^2]$, $[3.4.6n; 3.4.6n; 3.4.6n.4]$, 2 different series $[3.6^2; 6^2.2n; 3^2.6^2]$, $[4^3; 4^2.4n; 4^3.4n]$, $[4^2.6; 4^2.6; 4^2.6.2n]$, $[4^2.6; 4^2.6; 4.6.4.2n]$, 2 different series $[4^2.6; 4.6.2n; 4^3.6]$, $[4^2.2n; 4.6^2; 4^3.6]$, $[4^2.4n; 4^2.4n; 4^4]$, $[3.8^2; 3.8.4n; 3.8.4n]$, 2 different series $[4^2.6; 4.6.4n; 4.6.4n]$, $[4^2.8; 4.8^2; 4.8.2n]$, $[4^2.6n; 4^2.6n; 4^2.6n]$, $[4.6^2; 4.6^2; 6^2.2n]$ and $[4.6^2; 4.6.2n; 6^3]$, $n = 1, 2, ...$.

Remark that for isometry group series $*nn$, nn, $*22n$ and $n*$, $n = 1, 2, ...$, the Delone classes of normal 3-isohedral tilings of the sphere were listed in [6].

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Section 1.2

Differential Equations

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On Some Solutions of the 3D Navier-Stokes Equations

Iurie Baltag

Abstract

In this paper, there are determined various solutions of the stationary Navier-Stokes equations, which describe the 3D flow of an incompressible liquid or gas, i.e., solutions containing the components of the velocity of flow – the functions u, v, w and the created pressure P. We mention that in the paper a series of exact solutions are obtained, in which the viscosity coefficient λ participates explicitly.

Keywords: stationary three-dimensional Navier-Stokes equations, exact solutions, method of separation of variables, viscosity, pressure, velocity of flow of a liquid or gas.
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1 Introduction

In the present paper, the Navier-Stokes equations are studied in the stationary three-dimensional case. In this case, the Navier-Stokes equations represent a system containing four differential equations contained in system (1) and equation (2). Until today, the examined problem has not been definitively solved. The complexity of the problem lies in the fact that the equations in the system (1) are non-linear. A method is not developed that would allow us to determine all the solutions of this system. Determining the solutions of the system of Navier-Stokes equations is an important mathematical problem and has various applications in fluid and gas mechanics.

The following systems of partial differential equations is examined:

$$\begin{cases} \dfrac{P_x}{\mu} + uu_x + vu_y + wu_z = \lambda\Delta u + F_x, \\ \dfrac{P_y}{\mu} + uv_x + vv_y + wv_z = \lambda\Delta v + F_y, \\ \dfrac{P_z}{\mu} + uw_x + vw_y + ww_z = \lambda\Delta u + F_z. \end{cases} \quad (1)$$

$$u_x + v_y + w_z = 0. \quad (2)$$

Here $P = P(x, y, z)$, $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, $F = F(x, y, z)$; $u_x = \dfrac{\partial u}{\partial x}$; $\Delta u = u_{xx} + u_{yy} + u_{zz}$; $x, y, z \in R$.

Systems (1), (2) describe the process of the stationary flow of a liquid or gas in three-dimensional space. The system consisting of (1), (2) represents the Navier-Stokes equations in the case of the three-dimensional stationary motion of a viscous incompressible fluid or gas.

The function P represents the pressure of the liquid, and the functions u, v, w represent the flow velocity component of the liquid or gas. The function F represents the external force and is potential in nature. The constants $\lambda > 0$ and $\mu > 0$ are a determined parameter of the studied liquid's or gas viscosity and density. We mention here that $\lambda = c/R_e, c > 0$, where R_e is the Reynolds number.

Regarding the derivation of the equations of system (1), (2) and the meaning of the physical processes described by this system, consult the works [1], [2]. A number of solutions to the equations in the two-dimensional case have been determined in the papers [3], [4].

Regarding the application of various methods for determining the solutions of linear and nonlinear equations with partial derivatives, you can see the works [6].

2 Determination of solutions that do not explicitly contain the viscosity coefficient

The following theorem is true:

Theorem 1. *Let, in the connected domain D, the functions $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, and $P(x,y,z)$ admit continuous partial derivatives up to and including the second order. If in this domain the functions u,v,w, and P verify the following equalities:*

$$\begin{aligned} u_y &= v_x, u_z = w_x, w_y = v_z, u_x + v_y + w_z = 0, \\ P &= \mu\left[F + C - 0,5\left(u^2 + v^2 + w^2\right)\right], \end{aligned} \qquad (3)$$

then these functions are the solutions of systems (1), (2) for any constant C.

Proof. Adding to both sides of the first equation of system (1) the expression $0,5\left(v^2 + w^2\right)_x$, in the second equation the expression $0,5\left(u^2 + w^2\right)_y$ and in the third equation of this system the expression $0,5\left(u^2 + v^2\right)_z$, we obtain the following system, which is equivalent to system (1):

$$\begin{cases} G_x = \lambda\Delta u - v\left(u_y - v_x\right) + w\left(w_x - u_z\right), \\ G_y = \lambda\Delta v + u\left(u_y - v_x\right) - w\left(v_z - w_y\right), \\ G_z = \lambda\Delta w - u\left(w_x - u_z\right) + v\left(v_z - w_y\right), \end{cases} \qquad (4)$$

where $G = \mu^{-1}P - F + 0,5\left(u^2 + v^2 + w^2\right)$.

If in the connected domain D the functions u, v, w, and P admit continuous partial derivatives up to and including the second order and $u_y = v_x$, $u_z = w_x$, $w_y = v_z$, $u_x + v_y + w_z = 0$, then $\Delta u = \Delta v = \Delta w = 0$. Therefore, from the system (4), it follows that in this domain, the function G is a constant and the pressure is equal $P = \mu\left[F + C - 0,5\left(u^2 + v^2 + w^2\right)\right]$.

Theorem 1 is proved.

Below we present an example of solutions to the system (1), (2) that verify the conditions of Theorem 1.

Example 1 For the flow velocity components, we have $u = cxt^{-1,5}$, $v = cyt^{-1,5}$, $w = czt^{-1,5}$; $t = x^2 + y^2 + z^2$, c is arbitrary constant. The pressure P is determined from (3).

The following theorem generates a series of solutions of equations (1), (2) which can be obtained from formulas (3).

Theorem 2. *Let* $f = a(x)b(y)c(z)$, *where the functions* $a(x)$, $b(y)$, *and* $c(z)$ *are doubly differentiable and satisfy the following equations:*

$a^{''} = ra, b^{''} = sb, c^{''} = lc; r + s + l = 0$, *here* r, s, l *are constants.*

Then for the flow velocity components of the problem (1), (2) we have $u = f_x, v = f_y, w = f_z$. *The pressure* P *is determined from (3).*

Proof. Under the conditions of Theorem 2, the function f has continuous partial derivatives up to the second order inclusive. Because $u = f_x$, $v = f_y$, $w = f_z$, then the following equalities $u_y = v_x$, $u_z = w_x$, $w_y = v_z$ are valid. From condition (2), we obtain that

$$a^{''}(x)b(y)c(z) + a(x)b^{''}(y)c(z) + a(x)b(y)c^{''}(z) = 0.$$

Dividing both sides of this equation by $a(x)b(y)c(z)$, we get the following equation: $\dfrac{a^{''}(x)}{a(x)} + \frac{b^{''}(y)}{b(y)} + \frac{c^{''}(z)}{c(z)} = 0 \Leftrightarrow a^{''} = ra$, $b^{''} = sb$, $c^{''} = lc$; $r + s + l = 0$; $r, s, l - const$. Thus, the functions u, v, w are determined. After this, it is determined the pressure P from (3).

Theorem 2 is proved.

Suppose that under the conditions of Theorem 2: $r > 0$, $l > 0$, $s = -(r + l) < 0$. Then from the equations $a^{''} = ra$, $b^{''} = sb$, $c^{''} = lc$, we will obtain:

$$a(x) = a_1 e^{\sqrt{r}x} + a_2 e^{-\sqrt{r}x}; b(y) = b_1 e^{\sqrt{l}y} + b_2 e^{-\sqrt{l}y},$$
$$c(z) = c_1 \cos\left(\sqrt{r + l}z\right) + c_2 \sin\left(\sqrt{r + l}z\right),$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ – arbitrary constants.

Then

$$f = \left(a_1 e^{\sqrt{r}x} + a_2 e^{-\sqrt{r}x}\right) \times \left(b_1 e^{\sqrt{l}y} + b_2 e^{-\sqrt{l}y}\right) \times$$
$$\times \left(c_1 \cos\sqrt{r + l}z + c_2 \sin\sqrt{r + l}z\right).$$

Analogously, we obtain f in the other cases with compliance with the condition $r + s + l = 0$.

Example 2 The components of velocity are: $u = 3c\sin(5z)e^{3x+4y}$, $v = 4c\sin(5z)e^{3x+4y}$, $w = 5c\cos(5z)e^{3x+4y}$, c is constant.

The pressure P is determined from (3).

3 The solutions that explicitly contain the viscosity coefficient λ

The following two theorems give solutions in which the viscosity parameter λ participates explicitly

Theorem 3. *Let $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t)$, $t = x + y + z$, where the functions $f_k, k = 1, 2, 3$ are differentiable up to and including the third order. Then*

$$u = \frac{-1}{3C}\left[(C_1 + C_3)\,t + C_2 + C_4 + 3\lambda C^{-1}(C_1 + C_3) + (C_0 + C_5)\,e^{\frac{Ct}{3\lambda}}\right],$$

$$v = \frac{1}{3C}\left[(2C_1 - C_3)\,t + 2C_2 - C_4 + 3\lambda C^{-1}(2C_1 - C_3) + \right.$$
$$\left. + \,(2C_0 - C_5)\,e^{\frac{Ct}{3\lambda}}\right],$$

$$w = C + \frac{1}{3C}\left[(2C_3 - C_1)\,t + 2C_4 - C_2 + 3\lambda C^{-1}(2C_3 - C_1) + \right.$$
$$\left. + \,(2C_5 - C_0)\,e^{\frac{Ct}{3\lambda}}\right].$$

where $C, C_0, C_1, C_2, C_3, C_4, C_5$ are constant.

For pressure we get:

$$P = \mu\left[F - 3\lambda\left(u'(t) + v'(t)\right) + uv + 0,5\left(u^2 + v^2 - w^2\right)\right]. \qquad (5)$$

Proof. Let $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t)$, $t = x + y + z$. From (2), we immediately deduce that $f_3' = -f_1' - f_2' \Rightarrow f_3 = C - f_1 - f_2$; C is some constant.

Then from system (4), we get:

$$\begin{cases} G_x = 3\lambda f_1'' - f_2(f_1' - f_2') + (C - f_1 - f_2)(-2f_1' - f_2') = \\ 3\lambda f_1'' - C(2f_1' + f_2') + Q, \\ G_y = 3\lambda f_2'' + f_1(f_1' - f_2') - (C - f_1 - f_2)(2f_2' + f_1') = \\ 3\lambda f_2'' - C(2f_2' + f_1') + Q, \\ G_z = -3\lambda(f_1'' + f_2'') + f_1(2f_1' + f_2') + f_2(2f_2' + f_1') = \\ -3\lambda(f_1'' + f_2'') + Q. \end{cases} \tag{6}$$

Here $Q = (f_1 f_2)' + 2(f_1 f_1') + 2(f_2 f_2')$.

Since the functions $f_k, k = 1, 2, 3$ are differentiable up to and including the third order, system (6) will have solutions in any connected domain if and only if the following conditions $G_{xy} = G_{yx}$, $G_{xz} = G_{zx}$, $G_{yz} = G_{zy}$ are met. Applying these conditions, we arrive at a system in which only the functions f_k participate:

$$\begin{cases} 3\lambda(f_1'' - f_2'')' - C(f_1' - f_2')' = 0, \\ 3\lambda(2f_1'' + f_2'')' - C(2f_1' + f_2')' = 0, \\ 3\lambda(f_1'' + 2f_2'')' - C(f_1' + 2f_2')' = 0. \end{cases}$$

We notice that by subtracting from the second equation the third equation of the given system, we will obtain exactly its first equation. Therefore, we omit the last equation of the system, and we integrate the first two equations twice or consecutively. Thus, we arrive at a system equivalent to system (6):

$$\begin{cases} 3\lambda\,(f_1 - f_2)' - C\,(f_1 - f_2) = C_1 t + C_2, \\ 3\lambda\,(2f_1 + f_2)' - C\,(2f_1 + f_2) = C_3 t + C_4, \end{cases} \tag{7}$$

where $t = x + y + z$, C, C_1, C_2, C_3, C_4 are arbitrary constants.

Solving system (7), we will obtain:

$$\begin{cases} f_1 - f_2 = -C^{-1}(C_1 t + C_2 + 3\lambda C^{-1} C_1 + C_0 e^{\frac{C_t}{3\lambda}}), \\ 2f_1 + f_2 = -C^{-1}(C_3 t + C_4 + 3\lambda C^{-1} C_3 + C_5 e^{\frac{C_t}{3\lambda}}). \end{cases} \tag{8}$$

Solving system (8) by Cramer's method, we will obtain:

$$f_1 = \frac{-1}{3C}\left[(C_1+C_3)\,t + C_2 + C_4 + 3\lambda C^{-1}(C_1+C_3) + \right.$$
$$\left. + (C_0+C_5)\,e^{\frac{C_t}{3\lambda}}\right],$$
$$f_2 = \frac{1}{3C}\left[(2C_1-C_3)\,t + 2C_2 - C_4 + 3\lambda C^{-1}(2C_1-C_3) + \right.$$
$$\left. + (2C_0-C_5)\,e^{\frac{C_t}{3\lambda}}\right].$$

Here $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t) = C - f_1(t) - f_2(t)$.

The pressure P is determined from system (6). Theorem 3 is proved.

Example 3 The components of velocity are: $u = t + 3\lambda C^{-1} + e^{\frac{C_t}{3\lambda}} + c$, $v = -t - 3\lambda C^{-1} - 3e^{\frac{C_t}{3\lambda}} - 2c$, $w = C + 2e^{\frac{C_t}{3\lambda}} + c$; C, c are constant.

The pressure P is determined from system (5).

Theorem 4. *Let* $u = C_1 f(t)$, $v = C_2 f(t)$, $w = C_3 f(t)$; $t = ax + by + cz$, *where* a, b, c, C_1, C_2, C_3 *are constants and function* $f(t)$ *admits continuous derivatives up to and including the third order. If the following conditions* $aC_1 + bC_2 + cC_3 = 0$, $bC_1 \neq aC_2$, $aC_3 \neq cC_1$ *and* $cC_2 \neq bC_3$ *are valid, then* $f(t) = \dfrac{kt^2 + mt + h}{\lambda}$, *where* k, m, h *are arbitrary constant.*

Theorem 4 is proved analogously to Theorem 3.

4 Conclusions

In the current article, there are determined a lot of exact solutions of the stationary three-dimensional Navier-Stokes equations. We mention, that the solutions determined by Theorems 1 and 2 presented in example (1), (2) do not explicitly depend on the viscosity.

In the other solutions obtained in Theorems 3 and 4, the dependence of the solutions on the viscosity parameter is explicitly indicated.

The values of this constants can be determined based on initial conditions and boundary conditions of the physical problems examinates.

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Mathematical Modelling of Immune Response to Infectious Diseases with Ecological Factor

Yaroslav Bihun, Oleh Ukrainets

Abstract

The abstract generalizes G.I. Marchuk's mathematical model of the human immune system's response to infectious disease. The negative impact of environmental pollution, described by the Hutchinson equation, on the course of the disease is considered. Stationary solutions are found, their stability is investigated, and a medical interpretation of the obtained results is provided.

Keywords: immune response, infectious disease, mathematical model, delay, stationary solution

The immune response involves the production of specific objects (antibodies, $F(t)$), which are generated by a cascade of plasma cells $C(t)$. Antibodies are capable of neutralizing or destroying foreign materials (antigens), the amount $V(t)$ of which changes over time $t \geq t_0 = 0$.

An effective mathematical model of the immune response to infectious diseases was developed by G.I. Marchuk, and other researchers elaborated it in numerous works [1], [2]. The course of infectious diseases, such as hepatitis and acute respiratory diseases, is influenced by factors such as chemical, radioactive, and other environmental pollutants [3]. The model takes into account an integral factor $E(t)$, which is the sum of m factors $E_i(t)$ and is represented as follows:

$$E(t) = a_1 E_1(t) + ... + a_m E_m(t),$$

where $a_i \geq 0$, $a_1 + ... + a_m = 1$.

Let's assume that the change of $E(t)$ occurs according to the generalized Hutchinson equation, which has the following form:

$$\frac{dE(t)}{dt} = r\left(1 - \left(\frac{E(t-\Delta)}{E^*}\right)^n\right)E(t), t > 0, \tag{1}$$

where $r > 0$ – coefficient of linear growth, $0 < \Delta$ – average time for the restoration of ecological balance, amount of which $E^* > 0, n > 0$.

It has been shown that the equilibrium state $E = E^*$ is locally asymptotically stable if

$$0 < rn\Delta < \pi/2. \tag{2}$$

The change over time of the factors V, E, F, C and the magnitude m, where $o \leq m(t) \leq 1$ is the extent of organ damage against which the antigen V is directed, is proposed to be described by a system of equations:

$$\begin{aligned} \frac{dV}{dt} &= (\beta - \gamma F)V, \\ \frac{dC}{dt} &= \alpha\xi(m)V_\tau F_\tau - \mu_c(C - C^*) - \varepsilon_c E, \\ \frac{dF}{dt} &= \rho C - (\mu_f + \eta\gamma V)F, \\ \frac{dm}{dt} &= \sigma V - \mu_m m + \varepsilon_m E, \end{aligned} \tag{3}$$

where $\xi(m) = 1$ for $m \in [0, m^*]$ and $\xi(m) = (m-1)/(m^*-1)$ for $m^* < m \leq 1$, having $m^* \in (0,1)$ and considering for $m \in [0, m^*]$ the immune system functions normally; $V_\tau(t) = V(t-\tau), F_\tau(t) = F(t-\tau)$.

Theorem 1. *If the coefficients in equations (1), (3) are non-negative and the initial functions* $V_0(t) \geq 0, F_0(t) \geq 0$ *for* $t \in [-\tau, 0]$ *and* $E_0(t) \geq 0$ *for* $t \in [-\Delta, 0]$ *, then there exists a unique non-negative solution to the initial value problem for the system of equations (1), (3).*

There always exists such a stationary solution

$$E_1 = E^*, V_1 = 0, C_1 = C^* - \frac{\varepsilon_c E^*}{\mu_c}, F_1 = \frac{\rho C_1}{\mu_f}, m_1 = \frac{\varepsilon_m E^*}{\mu_m} \tag{4}$$

that defines the state of a healthy organism under permissible environmental pollution levels. Let's perform a substitution in system (3): $E = \overline{E} + E^*, V = \overline{V}, C = \overline{C} + C_1, F = \overline{F} + F_1, m = \overline{m} + m_1$. The determinant of the corresponding linear system has the following form:

$$(\lambda + rn\Delta e^{-\lambda})(\beta - \gamma F_1 - \lambda)(\mu_c + \lambda)(\mu_f + \lambda)(\mu_m + \lambda) = 0.$$

From condition (2) and the positivity of the coefficients in the system (3), a theorem is derived:

Theorem 2. *If condition (2) and conditions*

$$\beta - \gamma F_1 < 0, C^* \mu_c > \varepsilon_c E^*, \varepsilon_m E^* \le \mu_m m^*$$

are satisfied, then solution (4) is locally asymptotically stable.

Corollary 2.1. *When the conditions of Theorem 2 are met and $V_0 = V^*$, the disease will not occur. In this case, the stability depends on the delay Δ and is independent of the time $\tau > 0$ – formation of plasma cells. An estimate for V^* has been constructed.*

The system of equations (1), (3) may have another stationary solution that corresponds to the state of a chronic disease:

$$\begin{aligned} E_2 &= E^*, F_2 = \frac{\beta}{\gamma}, \\ V_2 &= \frac{\mu_c \mu_f \beta - \rho\gamma\mu_c C^* + \rho\gamma\varepsilon_c E^*}{\beta(\alpha\rho - \mu_c\eta\gamma)}, \\ C_2 &= \frac{\alpha\beta\mu_f - \eta\gamma^2\mu_c C^* + \eta\gamma^2\varepsilon_c E^*}{\gamma(\alpha\rho - \mu_c\eta\gamma)}, \\ m_2 &= \frac{\delta V_2 + E_2}{\mu_m}. \end{aligned} \tag{5}$$

A stationary solution (5) exists if either

$$\alpha\rho > \mu_c\eta\gamma, \quad \rho\gamma\mu_c C^* < \mu_c\mu_f\beta + \rho\gamma\varepsilon_c E^*$$

or the inequality with the opposite sign is satisfied.

From the analysis of the roots of the characteristic equation of the linearized system, conditions for the asymptotic stability and instability of solution (5) have been found. Therefore, sufficient conditions for either maintaining a chronic disease state or transitioning from a chronic condition to an acute form have been obtained.

Numerical simulations of the dynamics of the mathematical model (3) have been conducted.

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Polynomial Differential Cubic Systems with Invariant Straight Lines of Total Multiplicity Seven and Four Distinct Infinite Singularities

Cristina Bujac, Nicolae Vulpe

Abstract

The family of cubic differential systems possessing invariant straight lines was considered by many authors. But a strong classification of this family according to the configurations of invariant lines began in 2006 [11], where the existence of the maximum number of invariant lines (nine) is required. Here we continue the investigation of the subfamily of cubic systems possessing invariant lines of total multiplicity seven and four distinct infinite singularities. We prove that this subfamily could have a total of 166 distinct configurations of invariant straight lines.

Keywords: cubic system, affine transformation, invariant straight line, infinite and finite singularities, multiplicity of an invariant line and singularity, configuration of invariant straight lines.

1 Introduction and preliminary results

Consider real polynomial differential cubic systems

$$\frac{dx}{\partial t} = P(x, y), \quad \frac{dy}{\partial t} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, i.e., are polynomials in x, y with real coefficients and $\max(\deg P, \deg Q) = 3$.

Systems (1) depend on 20 parameters and their study is difficult in general, even impossible at this moment. For this reason, people began by studying particular subclasses of cubic systems, especially according to the geometrical proprieties of such systems and using the Qualitative Theory for their study.

Here we consider a particular case of cubic systems which possess invariant straight lines of total multiplicity seven. A line $f(x, y) = ux + vy + w = 0$ over C is an *invariant line* if and only if it there exists $K(x, y) \in \mathbb{C}[x, y]$, which satisfies the following identity in $\mathbb{C}[x, y]$: $uP(x, y) + vQ(x, y) = (ux + vy + w)K(x, y)$. We point out that if we have an invariant line $f(x; y) = 0$ over $\mathbb{C}[x, y]$, it could happen that multiplying the equation by a number $\lambda \in \mathbb{C}^*$, the coefficients of the new equation become real, i.e., $(u\lambda; v\lambda; w\lambda) \in \mathbb{R}^3$. In this case, along with the line $f(x, y) = 0$ sitting in $\mathbb{C}^2$, we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x, y) = 0$. Note that, since a system (1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux + vy + w = 0$, then its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$ is also invariant.

It is well known that the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [11], the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their configurations of invariant lines, where the authors detected 23 configurations. In [1], the author detected another class of cubic system whose configuration of invariant lines was not detected in [11]. Systems possessing 8 invariant straight lines, where the line at infinity is considered, have been made in [2], [3], [4], [5] and 51 distinct configurations have been detected. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [12].

Definition 1 [12]. *Consider a real planar polynomial differential system* (1). *We call configuration of invariant straight lines of this system, the set of (complex) invariant straight lines (which may have real coefficients) including the line at infinity of the system, each endowed* *with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.*

We continue our investigation of the family of cubic systems with invariant lines of total multiplicity seven considering the line at infinity (CSL_7). Moreover, we focus on such systems which have at infinity four singularities, all distinct.

It is well known that the infinite singularities (real or complex) of a system (1) are determined by the linear factors of the factorization of the cubic polynomial $C_3(x, y) = yp_3(x, y) - xq_3(x, y)$, where p_3 and q_3 are cubic homogeneities of systems (1). Clearly, for ex., in the case of four distinct infinite singularities, the polynomial $C_3(x, y)$ has four distinct linear factors, which could be all real (denoted in this case as $4r\infty$), two real and two complex ($2r2c\infty$), or all complex ($4c\infty$).

As we have four infinite singularities and the total multiplicity of invariant lines (including the line at infinity) must be 7, the systems (1) could only have one of the following five types of configurations of invariant lines – τ: $(3, 3); (3, 2, 1); (3, 1, 1, 1); (2, 2, 2)$ and $(2, 2, 1, 1)$. The notion of the type of configuration was introduced in [8].

In [7, 6, 9, 10], there are determined all possible configurations for the systems (1) belonging to the classes $CSL_\tau^{4r\infty}$ and $CSL_\tau^{2r2c\infty}$. Additionally, all the orbit representatives of the systems in $CSL_\tau^{4r\infty}$ and $CSL_\tau^{2r2c\infty}$ with respect to affine group of transformations and a time rescaling are obtained.

2 Main results

Notation 1 *We shall denote by* $\boldsymbol{CSL_7^{4s.p.\infty}}$ *the subfamily of cubic systems with invariant straight lines of total multiplicity 7 which have four distinct infinite singularities.*

We set the following notations related to configurations of invariant lines for a cubic system: $\mathcal{N}$ = the number of all distinct invariant straight lines (including the line at infinity); $\mathcal{M}_L$=the maximum multiplicity of an invariant line; $\mathcal{M}_s$=the maximum multiplicity of a real singularity located on the configuration. We apply the parameter $\mathcal{N}$ to split the set of all configurations according to the theorem given below.

According to [7, 6, 9, 10], the next result is obtained.

Theorem 1. *The subfamily* $\boldsymbol{CSL}_7^{4s.p.\infty}$ *has a total of 166 distinct configurations of invariant straight lines. More exactly:*

- *For* $\mathcal{N} = 7$*, we have* 109 *configurations among which there exist* 52 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_s = 2$.
- *For* $\mathcal{N} = 6$*, we have* 25 *configurations among which there exist* 11 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_L = 2$ *and* $\mathcal{M}_s = 6$.
- *For* $\mathcal{N} = 5$*, we have* 30 *configurations among which there exist* 13 *configurations having only real invariant lines. For this class, we have* $\mathcal{M}_L = 3$ *and* $\mathcal{M}_s = 9$.
- *For* $\mathcal{N} = 4$*, we have 2 configurations with only real invariant lines: one triple, one double, and two simple. In this class, we have* $\mathcal{M}_s = 6$.

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Two-sided Remotely Almost Periodic Solutions of Ordinary Differential Equations in Banach Spaces

David Cheban

Abstract

The aim of this paper is studying the two-sided remotely almost periodic solutions of ordinary differential equations in Banach spaces of the form $x' = A(t)x + f(t) + F(t,x)$ with two-sided remotely almost periodic coefficients if the linear equation $x' = A(t)x$ satisfies the condition of exponential trichotomy and nonlinearity F is sufficiently "small".

Keywords: Remotely Almost Periodic Solution; Non-autonomous Dynamical Systems; Cocycles.

1 Introduction

This study continues the author's work devoted to the study of remotely almost periodic (RAP) motions of dynamical systems and solutions of differential equations [2].

The notion of remotely almost periodicity on the real axis $\mathbb{R}$ for the scalar functions was introduced and studied by D. Sarason [7]. Remotely almost periodic functions on the semi-axis $\mathbb{R}_+$ with the values in the Banach space were introduced and studied by W. M. Ruess and W. H. Summers [6] (see also [2]). Remotely almost periodic functions on the real axis with the values in Banach spaces were introduced and studied by A. G. Baskakov [1]. He calls theses functions "almost periodic at infinity". Remotely almost periodic (on the real-axis $\mathbb{R}$) solutions of ordinary differential equations with remotely almost periodic coefficient were studied by C. Maulen, S. Castillo, M. Kostic, and M. Pinto [4].

2 Two-sided RAP Solutions of Linear Differential Equations

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $\mathbb{R} := (-\infty, +\infty)$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$. Denote by $[\mathfrak{B}]$ the Banach space of any linear bounded operators A.

Consider a linear differential equation

$$x' = A(t)x, \tag{1}$$

where $A \in C(\mathbb{R}, [\mathfrak{B}])$, and denote by $U(t, A)$ the Cauchy operator of the equation (1).

Definition 1. A linear differential equation (1) is said [3] to have an exponential trichotomy (on $\mathbb{R}$) if there exist linear projections P, Q such that $PQ = QP$, $P + Q - PQ = I$, and constants $K \geq 1$ and $\alpha > 0$ such that

$$\begin{gathered}
\|U(t,A)PU^{-1}(\tau,A)\| \leq Ke^{-\alpha(t-\tau)} \ \text{ for } \ 0 \leq \tau \leq t, \\
\|U(t,A)(I-P)U^{-1}(\tau,A)\| \leq Ke^{-\alpha(\tau-t)} \ \text{ for } \ t \leq \tau, \tau \geq 0 \\
\|U(t,A)QU^{-1}(\tau,A)\| \leq Ke^{-\alpha(\tau-t)} \ \text{ for } \ t \leq \tau \leq 0 \\
\|U(t,A)(I-Q)U^{-1}(\tau,A)\| \leq Ke^{-\alpha(t-\tau)} \ \text{ for } \ \tau \leq t, \ \tau \leq 0.
\end{gathered}$$

Definition 2. If the equation (1) has the exponential trichotomy on $\mathbb{R}$, then for (1) can be constructed a Green function [3, 5] by the equality

$$G_A(t,\tau) = \begin{cases} U(t,A)PU^{-1}(\tau,A), & (0 \leq \tau < t) \\ -U(t,A)(I-P)U^{-1}(\tau,A), & (t \leq \tau, \ \tau \geq 0) \\ -U(t,A)QU^{-1}(\tau,A), & (t \leq \tau \leq 0) \\ U(t,A)(I-Q)U^{-1}(\tau,A), & (\tau \leq t, \ \tau \leq 0). \end{cases} \tag{2}$$

Denote by $C(\mathbb{R}, \mathfrak{B})$ the space of all continuous functions $\varphi : \mathbb{R} \to \mathfrak{B}$ equipped with the compact-open topology.

Definition 3. A subset $A \subseteq \mathbb{R}$ is called relatively dense if there exists a positive number l such that $A \bigcap [a, a + l] \neq \emptyset$ for any $a \in \mathbb{R}$.

Definition 4. A function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ is said to be:

1. two-sided remotely almost periodic (respectively, remotely ω-periodic or remotely stationary) if for arbitrary $\varepsilon > 0$ there exists a relatively dense subset $\mathcal{F}(\varepsilon, \varphi)$ (respectively, $\omega\mathbb{Z} \subseteq \mathcal{F}(\varepsilon, \varphi)$ or $\mathbb{R} = \mathcal{F}(\varepsilon, \varphi)$) such that, for any $\tau \in \mathcal{F}(\varepsilon, \varphi)$, we can find a positive number $L(\varepsilon, \tau, \varphi)$ with the property $|\varphi(t + \tau) - \varphi(t)| < \varepsilon$ for any $|t| \geq L(\varepsilon, \tau, \varphi)$;

2. Lagrange stable if the subset $\{\varphi^h | \ h \in \mathbb{R}\}$ of $C(\mathbb{R}, \mathfrak{B})$ is precompact.

Theorem 5. Assume that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ and the following conditions are fulfilled:

1. (A, f) is Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

2. the equation (1) has an exponential trichotomy on $\mathbb{R}$.

Then

1. the equation
$$u' = A(t)u + f(t) \tag{3}$$
has a unique two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution φ_0 with $R\varphi_0(0) = 0$, where $R := PQ$;

2. the solution φ_0 is defined by $\varphi_0(t) = \int_{-\infty}^{+\infty} G_A(t, \tau) f(\tau) d\tau$, where $G_A(t, \tau)$ is the Green function of the equation (1) defined by (2);

3. every bounded on $\mathbb{R}$ solution of the equation (3) is two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary).

3 Two-sided RAP Solutions of Perturbed Linear Differential Equations

Let $W \subseteq X$ be a bounded (respectively, compact) subset of X. Denote by $C(\mathbb{R} \times W, X)$ the space of all continuous functions $f : \mathbb{R} \times W \to X$ equipped with the compact-open topology.

Definition 6. A function $F \in C(\mathbb{R} \times W, X)$ is said to be Lagrange stable if the set $\Sigma_F := \{\sigma(\tau, F) | \ \tau \in \mathbb{R}\}$ is pre-compact in $C(\mathbb{R} \times W, X)$.

Let us consider a differential equation

$$u' = A(t)u + f(t) + F(t, u), \tag{4}$$

where $F \in C(\mathbb{R} \times \mathfrak{B}, \mathfrak{B})$ and $F(t, 0) = 0$ for any $t \in \mathbb{R}$.

Assume that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is Lagrange stable and φ_0 is a unique remotely compatible solution of the equation (3) with $R\varphi_0(0) = 0$.

Theorem 7. Let $f \in C(\mathbb{R}, \mathfrak{B})$, $F \in C(\mathbb{R} \times \mathfrak{B}, \mathfrak{B})$ and $A \in (\mathbb{R}, [\mathfrak{B}])$. Assume that the following conditions are fulfilled:

1) the linear equation (1) has the exponential trichotomy on $\mathbb{R}$;

2) the functions A, f, and F are Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

3) F satisfies the condition of Lipschitz with respect to $x \in \mathfrak{B}$ with the constant of Lipschitz $L < \frac{\nu}{2\mathcal{N}}$.

Then

1. the equation (4) has at least one Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution φ defined by the equality

$$\varphi(t) = \int_{-\infty}^{+\infty} G_A(t, \tau)(f(\tau) + F(\tau, \varphi(\tau)))d\tau$$

and

2. $\|\varphi - \varphi_0\| \leq r$, where $r := \frac{4\mathcal{N}^2 L\|f\|}{\nu(\nu - 2\mathcal{N}L)}$.

Theorem 8. Let $A \in C(\mathbb{R}, [\mathfrak{B}])$, $f \in C(\mathbb{R}, \mathfrak{B})$ and $F \in C(\mathbb{R}_+ \times \mathfrak{B}, \mathfrak{B})$.

Assume that the following conditions are fulfilled:

1) the equation (1) has an exponential trichotomy;

2) the functions A, f, and F are Lagrange stable and two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary);

3) F is Lipschitzian with respect to the second variable with the constant $L > 0$.

Then the following statements hold:

1. there exists a number $\varepsilon_0 > 0$ such that, for every $|\varepsilon| \leq \varepsilon_0$, the equation
$$\frac{dx}{dt} = A(t)x + f(t) + \varepsilon F(t, x)$$
has a two-sided remotely almost periodic (respectively, remotely τ-periodic or remotely stationary) solution $\varphi_\varepsilon \in C_b(\mathbb{R}, \mathfrak{B})$ with $R\varphi_\varepsilon(0) = 0$;

2. $\{\varphi_\varepsilon\}$ converges to φ_0 as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}$.

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Darboux Integrability of a Cubic Differential System with One Invariant Straight Line and One Invariant Cubic

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Abstract

We find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one invariant straight line and one invariant cubic. The presence of a center at $O(0,0)$ is proved by method of Darboux integrability.

Keywords: cubic differential system, invariant algebraic curve, Darboux integrability, the problem of the center.

1 Introduction

We consider the cubic system of differential equations

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} &= -x - gx^2 - dxy - by^2 - sx^3 - qx^2y - nxy^2 - ly^3 \equiv Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$, $Q(x,y)$ are real and coprime polynomials in the variables x and y, $\dot{x} = dx/dt$, $\dot{y} = dy/dt$. The origin $O(0,0)$ is a singular point for (1) with purely imaginary eigenvalues, i.e., a focus or a center.

The problem of distinguishing between a center and a focus (the problem of the center) is open for general cubic differential system. It was solved for some families of cubic systems (1) with invariant algebraic curves: four invariant straight lines; three invariant straight lines [1]; two invariant straight lines and one invariant conic [1]; two invariant straight lines and one invariant cubic; two parallel invariant straight lines [2].

2 First integrals and integrating factors

It is known [1] that a singular point $O(0,0)$ is a center for (1) if and only if the system has in some neighborhood U of $O(0,0)$ a nonconstant analytic first integral $F(x,y)=C$ or an analytic integrating factor

$$\mu(x,y)=1+\sum_{k=1}^{\infty}\mu_k(x,y),$$

where μ_k are homogeneous polynomials of degree k.

An integrating factor for system (1) on some open set U of $\mathbb{R}^2$ is a C^1 function μ defined on U, not identically zero on U such that

$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\equiv 0. \qquad (2)$$

The relation $F(x,y)=C$ is a first integral of (1) on U if and only if

$$P(x,y)\frac{\partial F}{\partial x}+Q(x,y)\frac{\partial F}{\partial y}\equiv 0. \qquad (3)$$

Conditions for the existence of integrating factors of the form $\mu=\Phi^{\beta}$, $\beta\in\mathbb{R}$, were obtained for system (1) when $\Phi=0$ is an invariant cubic $a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3+x^2+y^2=0$ and when $\Phi=0$ is an invariant conic $a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1=0$ [3].

3 Darboux integrability

In this paper, we determine the center conditions for system (1) by constructing integrating factors and first integrals of the Darboux form

$$(1-x)^{\alpha_1}\Phi^{\alpha_2}, \qquad (4)$$

where $x=1$ is an invariant straight line and $\Phi(x,y)=0$ is an irreducible invariant cubic for (1); α_1,α_2 are real exponents, $\alpha_1\alpha_2\neq 0$.

Identities (2) and (3) will be used in finding integrating factors and first integrals of the form (4) for system (1). Identifying the coefficients in (2) ((3)) and solving the obtained algebraic system, we prove

Theorem 1. *The cubic system (1) has an integrating factor of the form (4), where* $\Phi(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2$, *if and only if one of the following nine sets of conditions holds:*

(i) $a = [(2b+3c+3)^2 + 24fu - 3u^2]/(8u)$, $d = [(2b+3c+3)^2 - 8fu + u^2]/(4u)$, $g = [(2b+3c+3)^3 + (2b+3c+3)(8fu-u^2) - 8u^2]/(8u^2)$, $k = -a$, $l = r = 0$, $m = -c-1$, $n = [(2b+3c+3)(u-12f)]/(4u)$, $p = -f$, $q = -d$, $s = [(2b+3c+3)(u^2-8fu-(2b+3c+3)^2)]/(8u^2)$, $4f(2b+3c+3) - u(2b+c+1) = 0$;

(ii) $a = 3f$, $c = (-2b-5)/3$, $d = [2f(2b+5)]/(1-2b)$, $g = (b-5)/3$, $k = -a$, $l = r = 0$, $m = (2(b+1))/3$, $n = -b$, $p = -f$, $q = -d$, $s = (2-b)/3$, $(2b-1)^2 - 108f^2 = 0$;

(iii) $b = 1/2$, $c = (-3)/2$, $d = (11a+21f)/3$, $g = -1$, $k = -a$, $l = f/2$, $m = 1/2$, $n = (27-84af-140a^2)/2$, $p = -f$, $q = (-2a-21f)/18$, $r = 0$, $s = (20a^2+12af+9)/36$, $(10a+6f)^2 - 27 = 0$;

(iv) $c = -1$, $d = -2a$, $f = -a$, $g = (2b-5)/2$, $k = -a$, $l = (-3a)/2$, $m = n = r = 0$, $p = a$, $q = (5a)/2$, $s = (3-2b)/2$;

(v) $b = 1/4$, $c = -2$, $d = -2a$, $f = -a$, $g = (-1)/4$, $k = -a$, $l = a/2$, $m = 1$, $n = (-1)/4$, $p = a$, $q = a/2$, $r = s = 0$;

(vi) $b = (2u^4 + 9uv - 9v^2 - 6au^3)/(18uv)$, $c = -(u^4 + 9uv + 9v^2 - 3au^3)/(9uv)$, $f = p = r = 0$, $g = (6au^2 - 9a^2u - u^3 - 3v)/(6v)$, $k = -a$, $m = -c-1$, $n = [(3a-u)(u-v)u]/(6v)$, $q = (9a^2u^2 - 6au^3 - 18av + u^4 + 9uv)/(18v)$, $s = [(9a^2u^2 - 6au^3 - 54av + u^4 + 9uv)(u-3a)u]/(162v^2)$, $u = 2a-d$, $v = 2a-d+6l$;

(vii) $c = 2(b-1)$, $d = -a$, $f = n = p = r = s = 0$, $g = (-1)/2$, $k = -a$, $l = -ab$, $m = 1-2b$, $q = a/2$;

(viii) $a = [-2(h^4l + h^3 + 8h^2l + 16l)]/[h^2(h^2+4)]$, $b = (-2h^4l + h^3 - 16h^2l - 32l)/[2h(h^2+4)]$, $c = (h^4l - h^3 + 8h^2l - 8h + 16l)/[h(h^2+4)]$, $d = -a + (6h)/(h^2+4)$, $f = p = r = 0$, $g = [-2(h^2+1)]/(h^2+4)$, $k = -a$, $m = -c-1$, $n = (3hl)/2$, $q = [-2(2h^4l - h^3 + 10h^2l + 8l)]/[(h^2+4)h^2]$, $s = (h^3 + 2h^2l + 8l)/[(h^2+4)h]$;

(ix) $c = 1-2b$, $d = 2(a+2f)$, $g = [(3a+4f)(1-b)]/(f)$, $k = -a$, $l = 2f$, $m = 2(b-1)$, $n = 4af - 3b + 4f^2 + 3$, $p = -f$, $q = 8(1-b)(a+f)$, $r = 0$, $s = [(4ab - 3a + 4bf - 4f)(b-1)]/(f)$.

Theorem 2. *The cubic system (1) has a first integral of the form (4), where* $\Phi(x,y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}$*, if and only if one of the following two sets of conditions holds:*

(x) $a = d = k = r = 0$, $g = c + 1 - b$, $l = bf$, $m = -(c+1)$, $n = [b(2c+3-b)]/2$, $p = -f$, $q = -f(c+1+b)$, $s = -b(c+1)$, $b^2 - 2f^2 - b = 0$;

(xi) $d = 2a$, $k = -a$, $l = [f(2b-c-1)]/3$, $m = -(c+1)$, $n = [(2b-c-2)(c+1)]/2$, $p = -f$, $q = a(2b-c-3)$, $r = 0$.

The existence of Darboux first integrals (integrating factors) of the form (4) implies the origin $O(0,0)$ to be a center for system (1).

Theorem 3. *The origin is a center for cubic system* (1) *with the invariant straight line* $1 - x = 0$ *and an invariant cubic* $\Phi(x,y) = 0$ *if one of the conditions* (i)–(xi) *holds.*

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The Riemann Spaces in the Theory of the Navier-Stokes Equations

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Abstract

A geometric approach for integrating the system of Navier-Stokes equations in the Euler and Lagrange variables is proposed.

Keywords: space of the Riemann, tensor Ricci, 6D-metric, Invariants, geodesics, incompressibility.

1 Introduction

The basis of the geometric approach for studying properties of Navier-Stokes system of equations (NS-system) composes the relations $x = f_1(a,b,c,t)$, $y = f_2(a,b,c,t)$, $z = f_3(a,b,c,t)$, $p = h(a,b,c,t)$, where (x,y,z,p) are the local coordinates for variable of pressure p and the variables (a,b,c) – for dual coordinates of individual points of the liquid at each moment of the parameter time t. As a result of elimination of the variables (a,b,c) with the help of the appropriate differentiations, the well-known system of Navier-Stokes equations in Euler variables is obtained

$$\frac{\partial}{\partial\eta}\vec{U} + (\vec{U}\cdot\vec{\nabla})\vec{U} + -\frac{1}{\rho}\vec{\nabla}P - \mu\Delta\vec{U} = 0, \quad div\ \vec{U} = 0, \tag{1}$$

where $\vec{U}$ is the vector of velocity of fluid, ρ is density, and μ is viscosity of fluid.

Similarly, after excluding of the coordinates (x,y,z) from the indicated relations, a system of equations, determining the movement of individual points of the liquid is obtained.

2 Ricci-flat metric associated with the NS-system

The $14D$-space in local coordinates $(x, y, z, t, \eta, \rho, m, u, v, w, p, \xi, \chi, n)$ is considered.

Theorem 1. *The metric of the form*

$$ds^2 = 2\,dx^2 + 2\,dxdy + 2\,dxdu + 2\,dy^2 + 2\,dydz + 2\,dydv + \\ 2\,dz^2 + 2\,dzdw + 2\,dtdp + 2\,d\eta d\xi + 2\,d\rho d\chi + 2\,dmdn + A dt^2 + \\ B d\eta^2 + C d\rho^2 + E dm^2, \tag{2}$$

where

$$A = 2 - U\,(x, y, z, t)\,u - V\,(x, y, z, t)\,v - W\,(x, y, z, t)\,w,$$

$$B = \left(-UW + \mu\,\frac{\partial}{\partial z}U\right) w + \left(-UV + \mu\,\frac{\partial}{\partial y}U\right) v + \\ \left(\mu\,\frac{\partial}{\partial x}U - (U)^2 - P\right) u - Up, \tag{3}$$

$$C = \left(-VW + \mu\,\frac{\partial}{\partial z}V\right) w + \left(\mu\,\frac{\partial}{\partial y}V - (V)^2 - P\right) v + \\ \left(-UV + \mu\,\frac{\partial}{\partial x}V\right) u - Vp, \tag{4}$$

$$E = \left(-\mu\,\frac{\partial}{\partial x}U - \mu\,\frac{\partial}{\partial y}V - (W)^2 - P\right) w + \left(-VW + \mu\,\frac{\partial}{\partial y}W\right) v + \\ \left(-UW + \mu\,\frac{\partial}{\partial x}W\right) u - Wp, \tag{5}$$

is the Ricci-flat on solutions of Navier-Stokes system of equations.

In the Lagrange variables (a, b, c, t), full system of Navier-Stokes equations looks moore complicated and has the form

$$\frac{\partial^2 A}{\partial t^2} + [B, C, P)] - \\ \mu\left([B, C, [B, C, A_t]] + [C, A, [C, A, A_t]] + [A, B, [A, B, A_t]]\right) = 0, \tag{6}$$

$$\frac{\partial^2 B}{\partial t^2} + [C, A, P] - \mu\left([B, C, [B, C, B_t]] + [B, A, [C, A, B_t]] + [A, B, [A, B, B_t]]\right) = 0, \tag{7}$$

$$\frac{\partial^2 C}{\partial t^2} + [A, B, P] - \mu\left([B, C, [B, C, C_t]] + [C, A, [C, A, C_t]] + [A, B, [A, B, C_t]]\right) = 0, \tag{8}$$
$$[A, B, C] - 1 = 0.$$

3 6D-metric

Theorem 2. *The* $6D$*-Riemann space in coordinates* (x, y, z, a, b, c, t)*, equipped with a metric of the form*

$$ds^2 = A(a, b, c, t)\, dx^2 + 2\, B(a, b, c, t)\, dxdy + 2\, E(a, b, c, t)\, dxdz + dxda + C(a, b, c, t)\, dy^2 + 2\, H(a, b, c, t)\, dydz + dydb + F(a, b, c, t)\, dz^2 + dzdc \tag{9}$$

can be used for integration of the equation

$$\frac{\partial}{\partial b}B\frac{\partial}{\partial c}C\frac{\partial}{\partial a}A - \frac{\partial}{\partial b}B\frac{\partial}{\partial a}C\frac{\partial}{\partial c}A - \frac{\partial}{\partial c}C\frac{\partial}{\partial a}B\frac{\partial}{\partial b}A - \frac{\partial}{\partial b}C\frac{\partial}{\partial c}B\frac{\partial}{\partial a}A + \frac{\partial}{\partial b}C\frac{\partial}{\partial a}B\frac{\partial}{\partial c}A + \frac{\partial}{\partial c}B\frac{\partial}{\partial a}C\frac{\partial}{\partial b}A - 1 = 0, \tag{10}$$

which is the condition of incompressibility of liquid.

4 Projective duality to the theory of the second order ODE's

$$\frac{d^2}{dx^2}y(x) + a_1\left(\frac{d}{dx}y(x)\right)^3 + 3\, a_2\left(\frac{d}{dx}y(x)\right)^2 + 3\, a_3\frac{d}{dx}y(x) + a_4 = 0, \tag{11}$$

where $a_i = a_i(x, y)$.

The general integral of this equation has the form $H(x,y,a,b)=0$ among four variables $(x,y,a,b)=0$, from which follows a differential equation with respect to the variables a,b

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}b\,(a)=g\left(a,b,\frac{\mathrm{d}}{\mathrm{d}a}b\,(a)\right). \tag{12}$$

Theorem 3. *Two equations* $y''=-a_1(x,y)y'^3-3a_2(x,y)y'^2-3a_3(x,y)y'-a_4(x,y)$ *and* $b"=g(a,b,b')$*, where* $b'=c$ *form a dual pair of equations with a common integral of the form* $H(x,y,a,b)=0$*, which with the help of solutions of the equation*

$$h\,(a,b,c)=\frac{\partial^2}{\partial c\partial a}g\,(a,b,c)+g\,(a,b,c)\,\frac{\partial^2}{\partial c^2}g\,(a,b,c)-1/2\,\left(\frac{\partial}{\partial c}g\,(a,b,c)\right)^2$$
$$+c\frac{\partial^2}{\partial c\partial b}g\,(a,b,c)-2\,\frac{\partial}{\partial b}g\,(a,b,c)$$
$$\frac{\partial^2}{\partial c\partial a}h\,(a,b,c)+g\,(a,b,c)\,\frac{\partial^2}{\partial c^2}h\,(a,b,c)-\left(\frac{\partial}{\partial c}g\,(a,b,c)\right)\frac{\partial}{\partial c}h\,(a,b,c)+$$
$$c\frac{\partial^2}{\partial c\partial b}h\,(a,b,c)-3\,\frac{\partial}{\partial b}h\,(a,b,c)=0, \tag{13}$$

is determined.

5 4D-metric

The geometric method for studying the properties of the NS-system of equations with the theory of second-order ODE's of normal projective connection of E. Cartan is associated.

Theorem 4. *Geodesic equations of the* $4D$*-space with the metric*

$$ds^2=(2\,za_3(x,y)-2\,ta_4(x,y))\,dx^2+2\,(2\,za_2(x,y)-2\,ta_3(x,y))\,dxdy+$$
$$2\,dxdz+(2\,za_1(x,y)-2\,ta_2(x,y))\,dy^2+2\,dydt, \tag{14}$$

in the coordinates (x, y, z, t) decomposes into two parts.

The first part consists of two equations:

$$\frac{d^2}{ds^2}y(s) + a_4(x,y)\left(\frac{d}{ds}x(s)\right)^2 + 2\,a_3(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) + a_2(x,y)\left(\frac{d}{ds}y(s)\right)^2 = 0, \tag{15}$$

$$\frac{d^2}{ds^2}x(s) - a_3(x,y)\left(\frac{d}{ds}x(s)\right)^2 - 2\,a_2(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s) - a_1(x,y)\left(\frac{d}{ds}y(s)\right)^2 = 0. \tag{16}$$

And the second part has the form of the linear system of equations for the vector $\vec{\Psi} = (\Psi_1 = z(s), \Psi_2 = t(s)$

$$\frac{d^2\vec{\Psi}}{ds^2} + A(x,y)\frac{d\vec{\Psi}}{ds} + B(x,y)\vec{\Psi} = 0, \tag{17}$$

where $\vec{\Psi} = (\Psi_1 = z(s),\ \Psi_2 = t(s))$ and $A(x,y), B(x,y)$ are the 2×2 matrix-functions.

Corresponding Ricci tensor R_{ij} of the metrics is of the form

$$\begin{aligned} R_{11} &= 2\,\frac{\partial}{\partial y}a_4(x,y) - 2\,\frac{\partial}{\partial x}a_3(x,y) - 4\,a_3(x,y)^2 + 4\,a_4(x,y)a_2(x,y), \\ R_{12} &= -2\,\frac{\partial}{\partial x}a_2(x,y) + 2\,\frac{\partial}{\partial y}a_3(x,y) - 2\,a_2(x,y)a_3(x,y) + 2\,a_4(x,y)a_1(x,y), \\ R_{22} &= -2\,\frac{\partial}{\partial x}a_1(x,y) + 2\,\frac{\partial}{\partial y}a_2(x,y) + 4\,a_1(x,y)a_3(x,y) - 4\,a_2(x,y)^2. \end{aligned} \tag{18}$$

6 Conclusion

This article proposes a geometric approach for constructing exact solutions of the Navier-Stokes equations and studying their properties.

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Weakly Hyperbolic IFS's with Condensation of Special Type

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Abstract

We show that for any weakly hyperbolic Iterated Function System with condensation in $\mathbb{R}^n$, whose condensation set is a union of a finite collection of convex compact sets, there exists a standard weakly hyperbolic IFS with the same attractor.

Keywords: Iterated Function System, weak contraction, attractor, convex set.

1 Introduction

The term fractal is usually associated with the attractor of a hyperbolic Iterated Function System (IFS) (see, e.g., [1]).

M. Barnsley [1] has introduced the idea of an Iterated Function System with condensation (IFSC), which means a hyperbolic IFS, accompanied by a constant compact-valued multi-function (condensation). This idea has led to new fractals as attractors of IFS's. However, the computer simulations of such IFS's create more problems than for standard hyperbolic IFS's.

For these reasons, and not only, the authors formulated some questions related to these topics (see [2, 3, 4, 5]):

1. Which compacta can (or can not) serve as attractors of hyperbolic IFS's?

2. Under which conditions the attractor of a hyperbolic IFS with condensation can be represented as the attractor of a standard hyperbolic IFS?

The second question is motivated, in particular, by the fact that for standard hyperbolic IFS's there are more tools to study and to justify various simulations by computer (e.g., the so-called "Chaos game") than for hyperbolic IFS's with condensation.

Moreover, the questions remain open for weakly hyperbolic IFS's as well.

One of the authors has partially answered the first question above for a special type of compact sets in the Euclidian space $\mathbb{R}^n$ (see [5]). He also answered the second question for the case when the condensation set is a compact convex set or a finite union of such sets in $\mathbb{R}^n$. Following these results, an algorithm to construct some type of plane fractals was proposed in [4, 6].

Here we purpose a generalization of these results to the case when the IFS consists of weakly contracting mappings and a condensation, whose condensation set is a compact convex set or a finite union of such sets. More precisely, we purpose a construction, which replaces the condensation mapping by a finite collection of contractions in such a way, that the attractor of the initial weakly hyperbolic IFS with condensation coincides with the attractor of the new (standard) weakly hyperbolic IFS.

2 Preliminaries

Let consider the Euclidean space $(\mathbb{R}^n, d)$.

Denote by $\mathcal{P}_{cp}(\mathbb{R}^n)$ the set of all nonempty compact subsets of $\mathbb{R}^n$, endowed with the Pompeiu-Hausdorff metric H.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [7, 8] if:

1) φ is monotonically increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

2) the iterations $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t \geq 0$.

It is known [8] that, if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.

Following [7, 8], we call a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ as a *weak contrac-*

tion, if there exists a comparison function φ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \qquad \forall\, x, y \in \mathbb{R}^n.$$

In this case, one says also that f is a *weak contraction with respect to the comparison function* φ or that it is a φ*-contraction* (see [8]).

Remark 1. For any finite collection of weak contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ there exists a common comparison function.

Recall that a *hyperbolic Iterated Function System (IFS)* is a finite collection of contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ $(1 \leq i \leq m)$, and it is denoted by $\{\mathbb{R}^n; f_1, \ldots, f_m\}$.

Similarly, a *weakly hyperbolic Iterated Function System (wIFS)* $\{\mathbb{R}^n; f_1, \ldots, f_m\}$ is a finite collection of weak contractions $f_i : \mathbb{R}^n \to \mathbb{R}^n$.

Associated with a (weakly) hyperbolic Iterated Function System $\mathcal{F} = \{\mathbb{R}^n; f_1, \ldots, f_m\}$, there is the mapping $F_* : \mathcal{P}_{cp}(\mathbb{R}^n) \to \mathcal{P}_{cp}(\mathbb{R}^n)$, defined by the equality $F_*(C) = \bigcup\limits_{i=1}^{m} \bigcup\limits_{x \in C} f_i(x)$, $C \in \mathcal{P}_{cp}(\mathbb{R}^n)$, and reffered to as the *Barnsley-Hutchinson operator*.

Remark 2. A weakly hyperbolic Iterated Function System is a particular case of a weakly contracting relation (see [9]), so we can apply all respective results.

Remark 3. Any weakly hyperbolic Iterated Function System, as a weakly contracting relation, admits a comparison function (e.g., a common comparison function for all components of wIFS).

Theorem 1. *Let* $\{X; f_1, \ldots, f_m\}$ *be a weakly hyperbolic IFS with respect to a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. *Then the corresponding Barnsley-Hutchinson operator* F_* *is also a* φ*-contraction, i.e., for any* $A, B \in \mathcal{P}_{cp}(\mathbb{R}^n)$, *the following inequality holds*

$$H(F_*(A), F_*(B)) \leq \varphi(H(A, B)).$$

A nonempty compact set $A \subset \mathbb{R}^n$ is said to be *attractor* of the weakly hyperbolic IFS $\mathcal{F}$, if $F_*(A) = A$, where F_* is the corresponding Barnsley-Hutchinson operator.

Theorem 2. *For any weakly hyperbolic IFS* $\{\mathbb{R}^n; f_1, \dots, f_m\}$*, there exists a unique compact set* A*, such that* $F_*(A) = A$*, where* F_* *is the corresponding Barnsley-Hutchinson operator.*

3 Sum of convex compacta as attractor of hyperbolic IFS

M. Barnsley (1988) introduced the concept of Iterated Function System with condensation.

A constant compact-valued function $f_0 : \mathbb{R}^n \to \mathcal{P}_{cp}(\mathbb{R}^n)$, $f_0(x) \equiv K$ for some $K \in \mathcal{P}_{cp}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, is called a *condensation* with K as *condensation set.*

Theorem 3. *[5] Any convex compact set in* $\mathbb{R}^n$ *can be represented as the attractor of a hyperbolic IFS, consisting only of contractions.*

Theorem 4. *[5] Given a finite family of convex compacta* $\{K_1, \dots, K_p\}$ *in* $\mathbb{R}^n$*, let* $\mathcal{F}_i = \{\mathbb{R}^n; \xi_{i1}, \dots, \xi_{iq_i}\}$ *stand for the corresponding hyperbolic IFS, having* K_i *as attractor.*

Then $K = \bigcup_{i=1}^{p} K_i$ *is the attractor of the hyperbolic IFS*

$$\mathcal{G} = \{\mathbb{R}^n; \psi_{11}, \dots, \psi_{1q_1}, \dots, \psi_{p1}, \dots, \psi_{pq_p}\},$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ *and* Pr_i *is the metric projection onto* K_i*.*

4 Attractors of weakly hyperbolic IFS with condensation

A *weakly hyperbolic Iterated Function System with condensation* (*wIFSC*) $\{X; f_0, f_1, \dots, f_m\}$ consists of a condensation f_0 and of certain weak contractions $f_1, \dots, f_m$.

The following result generalizes a similar one for hyperbolic IFS with condensation [5].

Theorem 5. *Let the compact set* $K \subset \mathbb{R}^n$ *be a union of convex compacta* $K = \bigcup_{i=1}^{p} K_i$, *and let* $\mathcal{F}_i = \{\mathbb{R}^n; \xi_{i1}, \ldots, \xi_{iq_i}\}$ *stand for the corresponding hyperbolic IFS, having* K_i *as attractor.*

Let $\mathcal{G} = \{\mathbb{R}^n; f_0, f_1, \ldots, f_m\}$ *be a weakly hyperbolic IFS with condensation, having* K *as the condensation set.*

Then the standard weakly hyperbolic IFS

$$\mathcal{T} = \{\mathbb{R}^n; \psi_{11}, \ldots, \psi_{1q_1}, \ldots, \psi_{p1}, \ldots, \psi_{pq_p}, f_1, \ldots, f_m\},$$

where $\psi_{ij} = \mathrm{Pr}_i \circ \xi_{ij}$ *and* Pr_i *is the metric projection onto* K_i, *has the same attractor as the given weakly hyperbolic IFS with condensation* $\mathcal{G}$.

5 Conclusion

These results show that in some conditions weakly hyperbolic IFS's with condensation can be studied by reducing them to standard weakly hyperbolic IFS's.

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The Stability Conditions of Unperturbed Motion Governed by Differential System with Nonlinearities of Fifth Degree

Natalia Neagu, Mihail Popa

Abstract

According to Lyapunov's first method of stability of unperturbed motion, there were determined the stability conditions for the differential system with nonlinearities of fifth degree.

Keywords: Differential systems, stability of unperturbed motion, center-affine transformation, characteristic equation.

1 Introduction

We will examine the two-dimensional differential system with nonlinearities of fifth degree of the form

$$\begin{aligned}\frac{dx}{dt} &= ax + by + ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5,\\ \frac{dy}{dt} &= cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5,\end{aligned} \tag{1}$$

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p$ are real arbitrary coefficients. Lyapunov proved that, for such systems, it can always be found a center-affine transformation, so that the system (1) contains a critical and a non-critical equation, that is

$$\begin{aligned}\frac{dx}{dt} &= ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5,\\ \frac{dy}{dt} &= cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5.\end{aligned} \tag{2}$$

2 The stability conditions of unperturbed motion governed by differential system with nonlinearities of fifth degree

We analyze the non-critical equation

$$cx + dy + kx^5 + 5lx^4y + 10mx^3y^2 + 10nx^2y^3 + 5oxy^4 + py^5 = 0. \quad (3)$$

Because the characteristic equation of the system (2) has one zero root and the other ones are real and negative, we have

$$I_1 = a^\alpha_\alpha = d < 0. \quad (4)$$

From the last relation, we express y

$$y = -\frac{c}{d}x - \frac{k}{d}x^5 - 5\frac{l}{d}x^4y - 10\frac{m}{d}x^3y^2 - 10\frac{n}{d}x^2y^3 - 5\frac{o}{d}xy^4 - \frac{p}{d}y^5. \quad (5)$$

We seek y as a holomorphic function of x. Then we can write

$$\begin{gathered} y = -\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + \\ +B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + B_{13}x^{13} + B_{14}x^{14} + \\ +B_{15}x^{15} + B_{16}x^{16} + ... \end{gathered} \quad (6)$$

Substituting (6) into (5), we have

$$\begin{gathered} -\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \\ +B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + B_{13}x^{13} + B_{14}x^{14} + B_{15}x^{15} + \\ +B_{16}x^{16} + ... = -\frac{c}{d}x - \frac{k}{d}x^5 - 5\frac{l}{d}x^4(-\frac{c}{d}x + B_2x^2 + B_3x^3 + B_4x^4 + \\ +B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + \\ +B_{13}x^{13} + B_{14}x^{14} + B_{15}x^{15} + B_{16}x^{16} + ...) - 10\frac{m}{d}x^3(-\frac{c}{d}x + B_2x^2 + \\ +B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + B_{10}x^{10} + B_{11}x^{11} + \end{gathered}$$

$$+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^2-10\frac{n}{d}x^2(-\frac{c}{d}x+$$
$$+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+$$
$$+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^3-$$
$$-5\frac{o}{d}x(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+$$
$$+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^4-\frac{p}{d}(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^5.$$

This implies that

$$B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+$$
$$+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...=$$
$$=-\frac{1}{d^6}(d^5k-5cd^4l+10c^2d^3m-10c^3d^2n+5c^4do-c^5p)x^5-$$
$$-\frac{5B_5}{d^5}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)x^9-\frac{5}{d^5}[2B_5^2d(d^3m-$$
$$-3cd^2n+3c^2do-c^3p)+B_9(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{13}-$$
$$-\frac{5}{d^5}[2B_5^3d^2(d^2n-2cdo+c^2p)+4B_5B_9d(d^3m-3cd^2n+3c^2do-c^3p)+$$
$$+B_{13}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{17}-$$
$$-\frac{5}{d^5}[B_5^4d^3(do-cp)+6B_5^2B_9d^2(d^2n-2cdo+c^2p)+4B_5B_{13}d(d^3m-$$
$$-3cd^2n+3c^2do-c^3p)+2B_9^2d(d^3m-3cd^2n+3c^2do-c^3p)+B_{17}(d^4l-$$
$$-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{21}-$$
$$-\frac{5}{d^5}[B_5^5d^4p+20B_5^3B_9d^3(do-cp)+30d^2B_5(B_5B_{13}+B_9^2)(d^2n-$$
$$-2cdo+c^2p)+$$
$$+20d(B_5B_{17}+B_9B_{13})(d^3m-3cd^2n+3c^2do-c^3p)+$$
$$+5B_{21}(d^4l-4cd^3m+6c^2d^2n-4c^3do+c^4p)]x^{25}+...$$

From this identity, we have

$$B_2 = B_3 = B_4 = 0,$$
$$B_5 = -\frac{1}{d^6}(d^5k - 5cd^4l + 10c^2d^3m - 10c^3d^2n + 5c^4do - c^5p),$$
$$B_6 = B_7 = B_8 = 0,$$
$$B_9 = -\frac{5B_5}{d^5}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p),$$
$$B_{10} = B_{11} = B_{12} = 0,$$
$$B_{13} = -\frac{5}{d^5}[2B_5^2d(d^3m - 3cd^2n + 3c^2do - c^3p) + B_9(d^4l - 4cd^3m+$$
$$+6c^2d^2n - 4c^3do + c^4p)], \quad B_{14} = B_{15} = B_{16} = 0,$$

$$B_{17} = -\frac{5}{d^5}[2B_5^3d^2(d^2n - 2cdo + c^2p) + 4B_5B_9d(d^3m - 3cd^2n+$$
$$+3c^2do - c^3p) + B_{13}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)],$$
$$B_{18} = B_{19} = B_{20} = 0,$$
$$B_{21} = -\frac{5}{d^5}[B_5^4d^3(do - cp) + 6B_5^2B_9d^2(d^2n - 2cdo + c^2p)+$$
$$+4B_5B_{13}d(d^3m - 3cd^2n + 3c^2do - c^3p) + 2B_9^2d(d^3m-$$
$$-3cd^2n + 3c^2do - c^3p) + B_{17}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)],$$
$$B_{22} = B_{23} = B_{24} = 0,$$
$$B_{25} = -\frac{5}{d^5}[B_5^5d^4p + 20B_5^3B_9d^3(do - cp) + 30d^2B_5(B_5B_{13} + B_9^2)(d^2n-$$
$$-2cdo + c^2p) + 20d(B_5B_{17} + B_9B_{13})(d^3m - 3cd^2n + 3c^2do - c^3p)+$$
$$+5B_{21}(d^4l - 4cd^3m + 6c^2d^2n - 4c^3do + c^4p)], ... \tag{7}$$

Substituting (6) into the right-hand sides of the critical differential equations (2), we get the following identity

$$ex^5 + 5fx^4y + 10gx^3y^2 + 10hx^2y^3 + 5ixy^4 + jy^5 = A_2x^2 + A_3x^3+$$

$$+A_4x^4+A_5x^5+A_6x^6+A_7x^7+A_8x^8+A_9x^9+A_{10}x^{10}+A_{11}x^{11}+A_{12}x^{12}+$$

$$+A_{13}x^{13}+A_{14}x^{14}+A_{15}x^{15}+A_{16}x^{16}+...,$$

or in detailed form

$$ex^5+5fx^4(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+$$
$$+B_{15}x^{15}+B_{16}x^{16}+...)+10gx^3(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+$$
$$+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+$$
$$+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^2+$$
$$+10hx^2(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+$$
$$+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^3+5ix(-\frac{c}{d}x+$$
$$+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+$$
$$+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+$$
$$+B_{16}x^{16}+...)^4+j(-\frac{c}{d}x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+$$
$$+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+$$
$$+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+B_{16}x^{16}+...)^5=A_2x^2+A_3x^3+$$
$$+A_4x^4+A_5x^5+A_6x^6+A_7x^7+A_8x^8+A_9x^9+A_{10}x^{10}+$$
$$+A_{11}x^{11}+A_{12}x^{12}+A_{13}x^{13}+A_{14}x^{14}+A_{15}x^{15}+A_{16}x^{16}+...$$

From here we obtain

$$A_2=A_3=A_4=0,$$
$$A_5=\frac{1}{d^5}(d^5e-5cd^4f+10c^2d^3g-10c^3d^2h+5c^4di-c^5j),$$
$$A_6=A_7=A_8=0,$$
$$A_9=\frac{5}{d^4}B_5(d^4f-4cd^3g+6c^2d^2h-4c^3di+c^4j),$$
$$A_{10}=A_{11}=A_{12}=0,$$
$$A_{13}=\frac{5}{d^4}[2B_5^2d(d^3g-3cd^2h+3c^2di-c^3j)+B_9(d^4f-4cd^3g+6c^2d^2h-$$

$$
\begin{gathered}
-4c^3di + c^4j)], \quad A_{14} = A_{15} = A_{16} = 0,\\
A_{17} = \frac{5}{d^4}[(2B_5^3d^2(d^2h - 2cdi + c^2j) + 4B_5B_9d(d^3g - 3cd^2h +\\
+3c^2di - c^3j) + B_{13}(d^4f - 4cd^3g + 6c^2d^2h - 4c^3di + c^4j)],\\
A_{18} = A_{19} = A_{20} = 0,\\
A_{21} = \frac{5}{d^4}[B_5^4d^3(di - cj) + 6B_5^2B_9d^2(d^2h - 2cdi + c^2j) +\\
+2d(2B_5B_{13} + B_9^2)(d^3g - 3cd^2h + 3c^2di - c^3j) + B_{17}(d^4f -\\
-4cd^3g + 6c^2d^2h - 4c^3di + c^4j)],\\
A_{22} = A_{23} = A_{24} = 0,\\
A_{25} = \frac{1}{d^4}[B_5^5d^4j + 20B_5^3B_9d^3(di - cj) + 30d^2B_5(B_5B_{13} +\\
+B_9^2)(d^2h - 2cdi + c^2j) + 20d(B_5B_{17} + B_9B_{13})(d^3g -\\
-3cd^2h + 3c^2di - c^3j) + 5B_{21}(d^4f - 4cd^3g + 6c^2d^2h -\\
-4c^3di + c^4j)], ...
\end{gathered} \tag{8}
$$

Let us introduce the following notations:

$$
\begin{gathered}
T = d^5k - 5cd^4l + 10c^2d^3m - 10c^3d^2n + 5c^4do - c^5p;\\
C = d^5e - 5cd^4f + 10c^2d^3g - 10c^3d^2h + 5c^4di - c^5j;\\
D = d^4f - 4cd^3g + 6c^2d^2h - 4c^3di + c^4j;\\
E = d^3g - 3cd^2h + 3c^2di - c^3j;\\
F = d^2h - 2cdi + c^2j; \quad G = di - cj.
\end{gathered} \tag{9}
$$

Then, taking into account (7), (8), and (9), we obtain

$$
\begin{gathered}
A_5 = \frac{C}{d^5}; \quad B_5 = -\frac{T}{d^6}; \quad A_9 = \frac{5}{d^4}B_5D,\\
A_{13} = \frac{5}{d^4}[2B_5^2dE + B_9D],\\
A_{17} = \frac{5}{d^4}[(2B_5^3d^2F + 4B_5B_9dE + B_{13}D];
\end{gathered}
$$

$$A_{21} = \frac{5}{d^4}[B_5^4 d^3 G + 6B_5^2 B_9 d^2 F + 2d(2B_5 B_{13} + B_9^2)E + B_{17}D];$$
$$A_{25} = \frac{1}{d^4}[B_5^5 d^4 j + 20B_5^3 B_9 d^3 G + 30d^2 B_5(B_5 B_{13} + B_9^2)F + \\ +20d(B_5 B_{17} + B_9 B_{13})E + 5B_{21}D], ... \tag{10}$$

Using the Lyapunov theorem [1], and the expressions (10), consi--dering that $I_1 = d < 0$, we get

Theorem 1. *The stability of unperturbed motion in the system of perturbed motion* (2) *is described by fourteen possible cases:*

I. $C > 0$, *then the unperturbed motion is stable*;

II. $C < 0$, *then the unperturbed motion is unstable*;

III. $C = 0$, $TD > 0$, *then the unperturbed motion is stable*;

IV. $C = 0$, $TD < 0$, *then the unperturbed motion is unstable*;

V. $C = D = 0$, $T \neq 0$, $E < 0$, *then the unperturbed motion is unstable*;

VI. $C = D = 0$, $T \neq 0$, $E > 0$, *then the unperturbed motion is stable*;

VII. $C = D = E = 0$, $TF > 0$, *then the unperturbed motion is stable*;

VIII. $C = D = E = 0$, $TF < 0$, *then the unperturbed motion is unstable*;

IX. $C = D = E = F = 0$, $T \neq 0$, $G > 0$, *then the unperturbed motion is stable*;

X. $C = D = E = F = 0$, $T \neq 0$, $G < 0$, *then the unperturbed motion is unstable*;

XI. $C = D = E = F = G = 0$, $Tj > 0$, *then the unperturbed motion is stable*;

XII. $C = D = E = F = G = 0$, $Tj < 0$, *then the unperturbed motion is unstable*;

XIII. $C = T = 0$, *then the unperturbed motion is stable*;

XIV. $e = f = g = h = i = j = 0$, *then the unperturbed motion is stable.*

In the last two cases, according to the Lyapunov theorem, the unperturbed motion belongs to some continuous series of stabilized motions. In this case, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotically stable [2] *in Cases* $I, III, VI, VII, IX,$ *and* XI. *The expressions* C, D, E, F, G, T *are from* (9).

Proof. According to Lyapunov's theorem [1], we analyze the coeffcients of the series (10).

Asume that $A_5 < 0$, then from (10) we get $\frac{C}{d^5} < 0$. Taking into account that $d < 0$, it follows $C > 0$. So we get case I, and similarly we get case II.

Suppose in (9) that $T \neq 0$. Then, from (10), we have $B_5 \neq 0$.

If $A_5 = 0$ $(C = 0)$, and $A_9 = \frac{5}{d^4} B_5 D = -\frac{5}{d^{10}} TD$, then the stability or the instability of unperturbed motion is determined by the sign of expression TD. Then, using Lyapunov's theorem, we proved the Cases III and IV.

If $A_5 = A_9 = 0$ $(C = D = 0)$, then $A_{13} = \frac{10}{d^{15}} T^2 E$. Taking into account that $d < 0$, the stability or the instability of unperturbed motion is determined by the sign of expression E. Then, using Lyapunov's theorem, we proved the Cases V and VI.

If $A_5 = A_9 = A_{13} = 0$ $(C = D = E = 0)$, then $A_{17} = -\frac{10}{d^{20}} T^3 F$, and the stability or the instability of unperturbed motion is determined by the sign of expression TF. Then, using Lyapunov's theorem, we proved the Cases VII and VIII.

If $A_5 = A_9 = A_{13} = A_{17} = 0$ $(C = D = E = F = 0)$, then $A_{21} = \frac{5}{d^{25}} T^4 G$. Taking into account that $d < 0$, the stability or the instability of unperturbed motion is determined by the sign of expression G. Then, using Lyapunov's theorem, we proved the Cases IX and X.

If $A_5 = A_9 = A_{13} = A_{17} = A_{21} = 0$ $(C = D = E = F = G = 0)$, then $A_{25} = -\frac{1}{d^{30}} T^5 j$, and the stability or the instability of unperturbed motion is determined by the sign of expression Tj. Then, using Lyapunov's theorem, we proved the Cases XI and XII.

If $C = T = 0$, then all A_k $(k \geq 5)$ are equal to zero. Then, using

Lyapunov's theorem, we proved the Case XIII.

If $T \neq 0$ a nd $C = D = E = F = G = j = 0$ ($e = f = g = h = i = j = 0$), then we obtain the Case XIV. Theorem 1 is proved.

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On Singular Operators along a Contour with Corner Points

Vasile Neagu

Abstract

In this paper, a method for constructing the symbol for singular integral operators in the case of a piecewise Lyapunov contour is proposed. In the definition of the symbol function, there are involved numbers characterizing the space in which the research is carried out, as well as the values of the corner points of the contour, which makes it possible to obtain formulas for calculating the essential norms of singular operators and conditions for the solvability of singular equations with a shift.

Keywords: singular operator, Banach algebras, piecewise Lyapunov contour, symbol, Noether conditions.

1 Introduction

A large number of works are devoted to singular integral operators and Riemann boundary value problems in the case of a Lyapunov contour. It is enough to point out the monograph by I. Gokhberg and N. Krupnik [1], which contains an extensive bibliography on this issue. In the works [2], [3], and others, it was shown that the presence of corner points on the integration contour affects some properties of singular operators. Obtaining Noether conditions, as a rule, leads to the concept of the operator symbol, first introduced by S. Mikhlin, and which turned out to be fruitful in many branches of mathematics, including the construction of the Noetherian theory of singular integral operators [4], [5].

2 Algebra $\mathcal{U}_{p\beta}$

Let us introduce the following notation. We denote by $L(\mathcal{B})$ the algebra of all linear bounded operators acting in a Banach space $\mathcal{B}$. Let $\mathcal{U}_{p\beta}$ be the smallest Banach subalgebra with algebra unit $L(L_p(R^+, t^\beta))$ $(R^+ = [0, +\infty))$, containing the operator

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\mathrm{R}^+} \frac{\varphi(\tau)}{\tau - t} d\tau \quad \left(t \in \mathrm{R}^+\right).$$

We will assume that $1 < p < \infty$ and $-1 < \beta < p-1$. Let δ be a number from the interval $\left(0, \frac{1}{2}\right)$. Let us denote by $l(\delta)$ an arc of a circle containing the points -1 and 1 and having the following property: from point z $(z \neq \pm 1)$ of the arc $l(\delta)$ the segment $[-1, 1]$ is visible at the angle of $2\pi\delta$, and when going around the arc $l(\delta)$ from point -1 to 1, this segment remains to the left. For numbers δ from the interval $\left(\frac{1}{2}, 1\right)$ we set $l(\delta) = -l(1-\delta)$. Let $l(\frac{1}{2})$ denote the segment $[-1, 1]$.

As is known [1], the spectrum of the operator S in the space $L_p(R^+, |t|^\beta)$ coincides with the arc $l\left(\frac{1+\beta}{p}\right)$. Since the algebra $\mathcal{U}_{p\beta}$ is generated by one element, then by [1] we have the following Theorem:

Theorem 1. *The set of maximal ideals of the algebra $\mathcal{U}_{p\beta}$ is homeomorphic to the arc $l = l\left(\frac{1+\beta}{p}\right)$. If M_z is the maximal ideal corresponding to the point $z(\in l)$, then the Gelfand transformation is $S((M_z) = z$.*

This Theorem can be significantly expanded (see [6]).

Theorem 2. *The algebra $\mathcal{U}_{p\beta}$ is an algebra without a radical with the symmetric involution $A \to \overline{A}$. In particular,*

$$\overline{S} = (\cos 2\pi\gamma S - i \sin 2\pi\gamma I)(\cos 2\pi\gamma I - i \sin 2\pi\gamma S)^{-1} \quad (\gamma = \frac{1+\beta}{p}).$$

For $p = 2$, the Gelfand transformation $A(z) = A(M_z)$ satisfies the equality

$$\|A\| = \max_{z \in l(\gamma)} |A(z)|, \tag{1}$$

and for $p \neq 2$, the following estimates hold:

$$\max_{z\in l(\gamma)} |A(z)| \leq \|A\| \leq c \cdot \max(\max_{z\in l(\gamma)} |A(z)|,$$

$$\max_{z\in l(\gamma)} \left| \left(1 - z^2\right) Ln \frac{1-z}{1+z} \frac{dA(z)}{dz} \right|), \tag{2}$$

where the constant c depends only on p and β.

Remark 1. Let us define the functional over $L_2(R^+, t^\beta)$ by the equality

$$f(\varphi) = \int_0^\infty \varphi(t) f(t) t^\beta dt,$$

then $S^* = t^{-\beta} S t^\beta I$. It is directly verified that $FBS^*B^{-1}F^{-1} = FB\overline{S}B^{-1}F^{-1}$. Therefore, for $p = 2$, we have $\overline{S} = S^*$.

In what follows, we will need the following Theorem.

Theorem 3. *Let $\omega = e^{\pi i \alpha}$, where α is some complex number. If $-1 < \operatorname{Re}\alpha < 1$, then the operator N_ω, defined by the equality*

$$(N_\omega \varphi)(x) = \frac{1}{\pi i} \int_{R^+} \frac{\varphi(y)}{y + \omega x} dy, \quad \left(x \in R^+\right),$$

belongs to the algebra $\mathcal{U}_{p\beta}$, and its Gelfand transformation has the form

$$N_\omega(z) = (z-1)^{\frac{1+\alpha}{2}} (z+1)^{\frac{1-\alpha}{2}} \quad (z \in l(\gamma)). \tag{3}$$

The branch of this function is chosen such that when $z = -ictg\pi\gamma$, it takes the value $-\frac{i\exp(-\pi i \gamma \alpha)}{\sin \pi\gamma}$.

3 Symbol of the operator $aI + bS_\Gamma$

Let the contour Γ consist of two semi-axes starting from the point $z = 0$. We denote by α $(0 < \alpha \leq \pi)$ the angle formed by these half lines. We will assume that one of these semi-straight lines coincides with the semi-axis $R^+ = [0,\ +\infty)$ and that the contour Γ is oriented such that on $\Gamma \bigcap R^+$ the orientation coincides with that on R^+.

Let $B = L_p\big(\Gamma,\ |t|^\beta\big)\ \ (-1 < \beta < p-1)$ and by $\lambda_0(\Gamma)$ we denote the set of piecewise constant functions on Γ that receive only two values: one value on R^+ and another value on $\Gamma\backslash R^+$. If $h \in \lambda_0(\Gamma)$, then we write

$$h\,(t) = \begin{cases} h_1, \ for \quad t \in R^+, \\ h_2, \ for \ \ t \in \Gamma\backslash R^+, \end{cases} \quad h_j \in \mathbb{C}.$$

So, $h\,(0) = h_2,\ h\,(0+0) = h_1,\ h\,(\infty - 0) = h_1,\ h\,(\infty + 0) = h_2.$

We will consider the contour Γ compactified with a point at infinity, whose neighborhoods are complementary to the neighborhoods of $z_0 = 0$. Obviously, the contour Γ is homeomorphic to a bounded contour $\widetilde{\Gamma}$, which has two angular points.

We denote by K_α the Banach algebra generated by the singular integration operator S_Γ and by all multiplication operators on the functions $h \in \lambda_0\,(\Gamma)$. By K^+ we denote the subalgebra of the algebra $L(L_p(R^+, |t|^\beta))$ generated by the singular integral operators $aI + bS$ $(S = S_{R^+})$ with constant coefficients on R^+. As K^+ is commutative, then it possesses [5] a sufficient system of multiplicative functionals. The operator ν,

$$(\nu\varphi)\,(x) = \big(\varphi\,(x)\,,\ \varphi\,(e^{i\alpha}x)\big) \quad \big(x \in R^+\big)\,,$$

is linear and bounded and acts from the space $L_p\big(\Gamma, |t|^\beta\big)$ to the space $L_p^2\left(R^+,\ t^\beta\right)$. Let $\varphi \in L_p\big(\Gamma, |t|^\beta\big)$, then the operator $\nu A \nu^{-1}$ has the form

$$\nu A \nu^{-1} = \left\| \begin{array}{cc} a_1 I + b_1 S & -b_1 M \\ b_2 N & a_2 I - b_2 S \end{array} \right\|, \tag{4}$$

where

$$(S\varphi)\,(t) = \frac{1}{\pi i}\int_{\mathrm{R}^+} \frac{\varphi\,(\tau)}{\tau - t}d\tau, \ \ (M\varphi)\,(t) = \frac{1}{\pi i}\int_{\mathrm{R}^+} \frac{\varphi\,(\tau)}{\tau - e^{-i\alpha}t}d\tau,$$

$$(N\varphi)(t) = \frac{1}{\pi i} \int_{\mathrm{R}^+} \frac{\varphi(\tau)}{\tau - e^{i\alpha}t} d\tau \quad (\ t \in \mathrm{R}^+).$$

From Theorems 2 and 3 it follows that the operators M and N belong to the algebra K^+ generated by the operator $S(= S_{R^+})$, and the multiplication operators – to the constant functions.

Therefore, $\nu K_\alpha \nu^{-1} \subset (K^+)^{2\times 2}$. Let $\{\gamma_M\}$ be the homeomorphism system that determines the symbol on the algebra K^+. For any operator $A \in K_\alpha$, we put

$$\widetilde{\gamma}_M(A) = \|\gamma_M(A_{jk})\|_{j,k=1}^2,$$

where $\|A_{jk}\|_{j,k=1}^2 = \nu A \nu^{-1}$.

4 Noetherian conditions

Theorem 4. *The operator $A \in K$ is Noetherian in the space $L_p\big(\Gamma,\ |t|^\beta\big)$ if and only if*

$$det\widetilde{\gamma}_M(A) \neq 0.$$

Conclusion 1. Theorem 4 allows us to define a symbol on the algebra K. Namely, it is natural to call the matrix $\widetilde{\gamma}_M(A)$ the symbol of the operators $A \in K$. Taking into account formulas (3) and (4), the symbol of the operators $H = hI \ \ (h \in \lambda_0(\Gamma))$ and S_Γ will have the form:

$$\widetilde{\gamma}_M(H) = \left\| \begin{matrix} h_1 & 0 \\ 0 & h_2 \end{matrix} \right\|,$$

$$\widetilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} z & (z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} \\ (z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} & -z \end{matrix} \right\|.$$

Remark 2. If $\alpha = \pi$, that is, the contour Γ satisfies the Lyapunov conditions at the point $z_0 = 0$, then the symbol of the operator $H = hI$ remains the same, and the symbol of the operator S_Γ has the form

$$\widetilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} z & -\sqrt{z^2-1} \\ \sqrt{z^2-1} & -z \end{matrix} \right\|,$$

$$\widetilde{\gamma}_M\left(S_\Gamma\right)=\left\|\begin{array}{cc} cth\pi\left(\xi+i\gamma\right) & \left(sh\pi(\xi+i\gamma)\right)^{-1} \\ \left(sh\pi(\xi+i\gamma)\right)^{-1} & -cth\pi\left(\xi+i\gamma\right) \end{array}\right\|.$$

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The Regular Perturbations of the Nonviscous Semilinear 1D Canh-Hilliard Equation

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Abstract

We study the behavior of solutions to the relaxation nonviscous semilinear one space dimension Canh-Hilliard equation when the relaxations tend to zero.

Keywords: Initial boundary problem, nonviscous semilinear Canh-Hilliard equation, regular perturbations.

1 Introduction

Let $I := (0, l) \subset \mathbb{R}$ and $L^2(I)$ be the usual real Hilbert space endowed with scalar product $(u, v) = \int_I u(x)\, v(x)\, dx$ and with the norm $||u||^2 = \int_I u^2(x)\, dx$. $H^k(I), k \in \mathbb{N}$ is the usual real Sobolev space endowed with the scalar product $(u, v)_{H^k(I)} = \int_I \sum_{s=0}^{k} u^{(s)}(x)\, v^{(s)}\, dx$. Denote by $V =: H_0^1(I)$ the closure of $C_0^\infty(I)$ in the norm of the space $H^1(I)$ which is endowed with the scalar product $(u, v) = \int_I u_x(x) \cdot v_x(x)\, dx$, and by $V' = H^{-1}(I)$ – the dual space for V. Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; X)$ the usual Sobolev spaces of the vectorial distributions $W^{k,p}(a, b; X) = \{f \in D'(a, b, X); f^{(l)} \in L^p(a, b; X), l = 0, 1, \ldots, k\}$ equipped with the norm $||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;X)} \Big)^{1/p}$.

Consider the following abstract initial boundary value problem for the relaxation Canh-Hilliard equation, which in what follows will be called (P_ε):

$$\begin{cases} \varepsilon\, u_\varepsilon''(t) + u_\varepsilon'(t) + A\big(Au_\varepsilon(t) + a\, u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t), \;\; t > 0, \\ u_\varepsilon\big|_{t=0} = u_{0\,\varepsilon}, \quad u_\varepsilon'\big|_{t=0} = u_{1\,\varepsilon}, \end{cases} \tag{P_ε}$$

where $A : D(A) = H^2(I) \bigcap H_0^1(I) \subset L^2(I) \mapsto L^2(I)$, $\quad A = -\frac{d^2}{dx^2}$. $B : D(B) \subseteq L^2(I) \to H^{-1}(I)$, $B(u) = b\,|u|^q\, u$, $u(t) =: u(\cdot, t)$, and by $'$ is denoted the derivative in t, $a, b \in \mathbb{R}$, and ε is a small positive parameter.

We will study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the unperturbed problem (P_0):

$$\begin{cases} v'(t) + A\,\big(Av(t) + a\, v(t) + B(v(t))\big) = f(t), \quad t > 0, \\ v\big|_{t=0} = u_0, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0$, where $u_{0\,\varepsilon} \to u_0$, $u_{1\,\varepsilon} \to u_1$ and $f_\varepsilon \to f$ in a certain sense, which will be specified below.

2 Existence and a *priori* estimates of solutions to the problems (P_ε) and (P_0)

We shall remind the definitions of solutions to the problems (P_ε) and (P_0) and also the existence theorems for solutions to these problems.

Definition 1. *Let $T > 0$, $f \in L^2(0,T; V')$ and $B : D(B) \subseteq L^2(I) \to V'$. We say a function $u_\varepsilon \in W^{2,2}\big(0,T; V'\big) \bigcap W^{1,2}\big(0,T; L^2(I)\big) \bigcap L^2\big(0,T; D(A) \bigcap D(B)\big)$ is a solution (quasi-strong solution) to the problem (P_ε) if u_ε satisfies the equation (P_ε) in V' a.e. $t \in (0,T)$ and the conditions from (P_ε).*

Definition 2. *Let $T > 0$, $f \in L^2(0,T; V')$ and $B : D(B) \subseteq L^2(I) \to V'$. We say a function $v \in L^2(0,T; V') \bigcap L^2(0,T; D(A) \cap$*

$D(B)$) is a solution to the problem (P_0) if v satisfies the equation (P_0) in V' a.e. $t \in (0,T)$ and the conditions from (P_0).

Using methods developed in [1], the following theorems are proved:

Theorem 1. *Let $T > 0$ and $q \geq 0$. If $u_{0\,\varepsilon} \in H^3 \bigcap V$, $u_{1\,\varepsilon} \in V$, and $f_\varepsilon \in L^2(0,T;V')$. Then there exits a unique solution $u_\varepsilon \in W^{2,\infty}\big(0,T;V'\big) \bigcap W^{1,\infty}\big(0,T;V\big) \bigcap L^\infty\big(0,T;H^3 \bigcap V\big)\big)$ to the problem (P_ε).*

Theorem 2. *Let $T > 0$ and $q \geq 0$. If $u_0 \in H^4(I) \cap V$ and $f \in W^{1,1}(0,T;L^2(I))$. Then there exits a unique solution to the problem (P_0), for which the estimate*

$$||v||_{C([0,t];L^2(I))} + ||v||_{L^2(0,t;H^2(I))} \leq C\,\tilde{M}_0(t), \quad \forall t \in [0,T],$$

is true with $C > 0$ and $\gamma > 0$, depending on a, b, q, l, and $\tilde{M}_0(t) = \Big(||u_0||_{H^2(I)} + ||f||_{L^2(0,t;L^2(I))}\Big)\, e^{\gamma\, t}$.

Lemma 1. *Suppose that $q \geq 0$, $b > 0$ and $\mu = 1 + a\left(\frac{l}{\pi}\right)^2 > 0$. If $u_{0\,\varepsilon} \in H^3 \bigcap V$, $u_{1\,\varepsilon} \in V$, $f_\varepsilon \in L^2_{loc}(0,\infty;L^2(I))$, then there exist the positive constants $C = C(a,b,q,l)$ such that for any solution u_ε to the problem (P_ε), the following estimate*

$$||u_\varepsilon(t)||_V^2 + \frac{b}{q+2}\,||u_\varepsilon(t)|^{q+2}_{L^{q+2}(I)} \leq C\,\mathcal{M}_0(t), \quad t \geq 0,\ \varepsilon \leq 1$$

is valid, where $\mathcal{M}_0(t) = ||u_{0\,\varepsilon}||_V^2 + ||u_{1\,\varepsilon}||^2 + ||A^{-1/2} f_\varepsilon||^2_{L^2(0,t;L^2(I))}$. If in addition $q \geq 1$, then there exist the positive constants $c_0 = c_0(b,q,\mu)$ and $C = C(b,q,\mu)$ such that

$$||u_\varepsilon(t)||_{H^2(I)} + ||u'_\varepsilon||_{L^2(0,t;I)} \leq C\,\mathcal{M}(t), \quad t \geq 0,\ \varepsilon \leq 1. \qquad (1)$$

$$\mathcal{M}(t) = ||u_{0\,\varepsilon}||_{H^2(I)} + ||u_{1\,\varepsilon}||_V + ||f_\varepsilon||_{L^2(0,t;L^2(I))} + \exp\{c_0 \int_0^t \mathcal{M}_0^{2q}(\tau)\,d\tau\}.$$

3 The relationship between solutions to the problems (P_ε) and (P_0)

Now, we will give the relationship between solutions to the problems (P_ε) and (P_0) in the linear case, which was inspired by the work [3]. For each $\varepsilon > 0$, the kernel $K(t,\tau,\varepsilon)$ of transformation which realizes this connection is the following:

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\,\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3\,K_2(t,\tau,\varepsilon) - 2\,K_3(t,\tau,\varepsilon)\Big),$$

where $\lambda(s) = \int\limits_s^\infty e^{-\eta^2}\,d\eta$, $\quad K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t-2\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t-\tau}{2\sqrt{\varepsilon\,t}}\Big)$,

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t+6\,\tau}{4\,\varepsilon}\Big\}\,\lambda\Big(\frac{2\,t+\tau}{2\sqrt{\varepsilon\,t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon\,t}}\Big).$$

The main properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 2 ([2]). *The function $K(t,\tau,\varepsilon)$ is a solution to the problem*

$$\begin{cases} K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t > 0, \quad \forall \tau > 0, \\ \varepsilon\,K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0 \\ K(0,\tau,\varepsilon = \dfrac{1}{2\,\varepsilon}\exp\Big\{-\dfrac{\tau}{2\,\varepsilon}\Big\}, \quad \forall \tau \geq 0, \end{cases}$$

from $C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$. This kernel possesses the following properties:

(i) $K(t,\tau,\varepsilon) > 0, \quad \displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,d\tau = 1, \quad \forall t \geq 0, \quad \forall \tau \geq 0;$

(ii) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2, and ε_0, all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}|t-\tau|^q\,d\tau \leq C_1\,e^{C_2 t}\,\varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t > 0;$$

Theorem 3. ([2]) *Suppose that* $e^{\gamma t} f_\varepsilon \in L^\infty(0,\infty; L^2(I))$, $e^{\gamma t} u_\varepsilon \in W^{2,\infty}(0,\infty; L^2(I)) \cap L^\infty(0,\infty; V)$ *and* $e^{\gamma t} Au_\varepsilon \in L^{2,\infty}(0,\infty; V')$, *where* u_ε *is a solution to the problem* (P_ε). *Then, for* $0 < \varepsilon < (4\gamma)^{-1}$, *the function* $w_\varepsilon(t) = \int\limits_0^\infty K(t,\tau,\varepsilon) u_\varepsilon(\tau) d\tau$ *is a solution to the problem*

$$\begin{cases} w'_\varepsilon(t)) + a\,Aw_\varepsilon(t) + A^2 w_\varepsilon(t) = F_0(t,\varepsilon), \quad \text{in } V', \quad \text{a.e.} \quad t \in [0,\infty), \\ w_{0\varepsilon} = \varphi_\varepsilon =: \int\limits_0^\infty e^{-\tau} u(2\,\varepsilon\,\tau) d\tau, \end{cases}$$

where $f_0(t,\varepsilon) = \dfrac{1}{\sqrt{\pi}}\Big[2\,\exp\Big\{\dfrac{3\,t}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\dfrac{t}{\varepsilon}}\Big) - \lambda\Big(\dfrac{1}{2}\sqrt{\dfrac{t}{\varepsilon}}\Big)\Big]$,

$F_0(t,\varepsilon) = f_0(t,\varepsilon)\, u_{1\varepsilon} + \int\limits_0^\infty K(t,\tau,\varepsilon)\, f_\varepsilon(\tau) d\tau.$

Moreover, $w_\varepsilon \in W^{1,\infty}_{\text{loc}}(0,\infty; L^2(\Omega)) \cap L^\infty_{\text{loc}}(0,\infty; V)$.

4 The main results

Now, let us present the main result contained in the following theorem.

Theorem 4. *Let* $T > 0$, $q \geq 1$, $b > 0$ *and* $\mu > 0$. *If* $u_0, u_{0\varepsilon} \in H^4(I) \cap V$, $u_{1\varepsilon} \in V$ *and* $f, f_\varepsilon \in W^{1,p}(0,T; L^2(I))(p \geq 1)$, *then there exist constants* C, c_0, *and* ε_0, *depending on* p, q, μ *and* T, *such that*

$$||u_\varepsilon - v||_{C([0,T];L^2(I))} \leq C\left[\mathcal{M}^{q+1}(T)\,\varepsilon^\beta + D_\varepsilon\right] e^{c_0\,\mathcal{M}^{2\,q}(T)}, \quad \varepsilon \in (0,\varepsilon_0],$$

where u_ε *and* v *are solutions to the problems* (P_ε) *and* (P_0), *respectively,*

$$D_\varepsilon = ||u_{0\varepsilon} - u_0|| + ||f_\varepsilon - f||_{L^2(0,T;L^2(I))}. \tag{2}$$

$\mathcal{M}(T)$ *is defined in (1) and* $\beta = \begin{cases} 1/2, & \text{if } f = 0, \\ (p-1)/(2\,p), & \text{if } f \neq 0. \end{cases}$

The proof of this result is based on the following two estimates, which are obtained using Theorem 3 as well as Lemmas 1, 2, and 3:

$$||u_\varepsilon(t) - w_\varepsilon(t)|| \leq C\, \mathcal{M}(T)\, \varepsilon^{1/2}, \quad t \in [0,T], \quad \varepsilon \in (0, \varepsilon_0],$$

$$||v(t) - w_\varepsilon(t)|| \leq C \left[\mathcal{M}^{q+1}(T)\, \varepsilon^{\beta} + D_\varepsilon \right] e^{c_0\, \mathcal{M}^{2\,q}(T)},\ t \in [0,T],\ \varepsilon \in (0, \varepsilon_0],$$

are true with $\mathcal{M}(T)$ from (1) and D_ε from (2).

5 Conclusion

The main result from Theorem 4 shows that the the small relaxations of IBV for nonviscous semilinear 1D space Canh-Hilliard equation are the regular character. Also we indicate a rate of speed of convergence of solutions of the perturbed problem to the corresponding solutions of the unperturbed problem when the relaxations tend to zero.

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Center Problem for Cubic Differential Systems with Two Affine Invariant Straight Lines of Total Transversal Multiplicity Tree

Alexandru Şubă

Abstract

In this paper, using the first five Lyapunov quantities, we solve the center problem for cubic differential systems with two affine invariant straight lines one of which has transversal algebraic multiplicity two.

Keywords: cubic differential system, the problem of the center, multiple invariant straight line.

1 Introduction

We consider the real cubic differential systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \gcd(P,Q) = 1, \end{cases} \quad (1)$$

and the vector fields $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$.

The *straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_{\mathcal{L}} \in \mathbb{C}[x,y]$ such that the identity $\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma) K_{\mathcal{L}}(x,y),\ (x,y) \in \mathbb{R}^2$, i.e., $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x,y) K_{\mathcal{L}}(x,y),\ (x,y) \in \mathbb{R}^2$, holds.

If $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$) is the greatest natural number such that $\mathcal{L}^{m_p(\mathcal{L})}$ (respectively, $\mathcal{L}^{m_a(\mathcal{L})}$) divides $\mathbb{X}(\mathcal{L})$ (respectively, $E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$), then we say that $\mathcal{L}$ has *parallel multiplicity* (*algebraic multiplicity*, or in brief, *multiplicity*) $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$). We have $1 \le m_p(\mathcal{L}) \le 3,\ 1 \le m_a(\mathcal{L}) \le 7$ and $m_p(\mathcal{L}) \le m_a(\mathcal{L})$. If $m_a(\mathcal{L}) > m_p(\mathcal{L})$, then we say that $\mathcal{L}$ has *transversally multiplicity* $m_t(\mathcal{L}) = m_a(\mathcal{L}) - m_p(\mathcal{L}) + 1$.

Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic systems with parallel multiple invariant straight lines (with the line at infinity and an affine real invariant straight line of total multiplicity four) were investigated in [3], [5] (in [6]).

Let $\mathcal{L}$ to be an invariant straight line for (1). A function $f = \exp\left(\frac{g}{\mathcal{L}^j}\right)$, $1 \le j \le m_a(\mathcal{L}) - 1$, $g \in \mathbb{C}[x,y]$, $deg(g) \le j$, is called *exponential factor*) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $\deg(K_f) \le 2$ such that the identity $\mathbb{X}(f) \equiv f \cdot K_f(x,y)$ holds.

The critical point $(0,0)$ of the system (1) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center.*

A critical point $(0,0)$ is a center for (1) if and only if the system has a nonconstant analytic first integral $F(x,y)$ (integrating factor $\mu(x,y)$) in a neighborhood of $(0,0)$. Also, $(0,0)$ is a center iff all Lyapunov quantities L_j vanish (see, [3]). If $F(x,y)$ $(\mu(x,y))$ has the form $f^{\alpha_1} \cdots f^{\alpha_\sigma}$, where f_j, $1 \le j \le \delta$ are invariant straight lines and f_j, $\delta \le j \le \sigma$ are exponential factors, then the system (1) is called Darboux integrable.

2 Condition of existence invariant straight lines

Denote $\kappa(x,y) = x(sx^3 + (q-a)x^2y + (n-c-1)xy^2 + (l-f)y^3)$.

Let $\mathcal{L}_1$, $\mathcal{L}_2$ be invariant straight lines for (1) and $m_a(\mathcal{L}_1) \ge 2$. The case $m_p(\mathcal{L}_1) \ge 2$ was investigated in [2].

Without lose of generality, we consider $\mathcal{L}_1 \equiv x - 1$. Suppose that $m_t(\mathcal{L}_1) \ge 2$. Then x^2 divides $\kappa(x,y)$. The direct calculus show that the straight lines $\mathcal{L}_1 \equiv x - 1 = 0$, $\mathcal{L}_2 \equiv \alpha x - y + \beta = 0, \beta \ne 0$ are invariant for (1) and $m_t(\mathcal{L}_1 \ge 2$ if and only if at least one of the following three series of conditions hold

$$\begin{array}{l} b=-(\alpha+f\alpha\beta+f\beta^2)/\beta, d=((\alpha+\beta)(\alpha-2\beta-c\beta)-1)/\beta,\\ m=-c-1, g=(\alpha-\beta-a\alpha\beta-a\beta^2)/\beta, k=-a, l=f,\\ n=(\alpha+2\beta+c\beta)/\beta, p=-f, q=(1-\alpha^2+a\beta-\alpha\beta)/\beta,\\ r=0, s=-\alpha/\beta; \end{array} \quad (2)$$

$$\begin{array}{l} a=-(\alpha+\beta)(2+c+f\alpha+f\beta), b=-(\alpha+f\alpha\beta+f\beta^2)/\beta,\\ r=0, g=(2f\alpha+(c+2)(1+d\beta)+(\alpha+\beta)(df\beta-(c+2)\cdot\\ (\alpha-2\beta-c\beta))-f(\alpha+\beta)^2(\alpha-4\beta-2c\beta)+f^2\beta(\alpha+\beta)^3)/\\ (f\beta), k=-a, l=f, n=-(1-\alpha^2+d\beta+c\alpha\beta)/\beta^2, p=-f,\\ q=-(df\beta^2+2\beta(d+\alpha+2\beta)+(c+2)(1-\alpha^2)+c\beta(d+3\alpha+\\ 4\beta)+\beta(\alpha+\beta)(c^2+4f\beta+2cf\beta)+f^2\beta^2(\alpha+\beta)^2)/(f\beta^2),\\ s=\alpha(f\alpha+2d\beta+2\beta(\alpha+2\beta)+(c+2)(1-\alpha^2)+c\beta(d+3\alpha+\\ 4\beta)-(\alpha+\beta)(f(\alpha-2\beta)(\alpha+\beta)-\beta(c^2+df))+\\ cf\beta(\alpha+\beta)^2)/(f\beta^2), m=-c-1; \end{array} \quad (3)$$

$$\begin{array}{l} a=-(2+c)(\alpha+\beta), d=((\alpha+\beta)(\alpha-2\beta-c\beta)-1)/\beta, f=0,\\ b=-\alpha/\beta, g=(\alpha+\alpha\beta(c+1)(\alpha+\beta)-q\beta^2)/\beta, k=-a, l=0,\\ m=-c-1, r=0, n=(\alpha+2\beta+c\beta)/\beta, p=-f,\\ s=-(\alpha(q\beta+(\alpha+\beta)(\alpha+2\beta+c\beta)))/\beta. \end{array} \quad (4)$$

3 Problem of the center

In each of the conditions (2) and (3) we calculate the first five Lyapunov quantities $L_1, ..., L_5$. Solving the system $\{L_1=0, L_2=0, ..., L_5=0\}$, we obtain for (1) all the cases of the center in the sets of conditions (2) and (3). Under the conditions (4) for solving the center problem, it is necessary and sufficient as the first three Lyapunov quantities to vanish. In this way, the following Theorem is valid.

Theorem 1. *The cubic system* (1) *with two affine invariant straight lines of total transversal multiplicity three has at origin* $(0,0)$ *center if and only if the first five Lyapunov quantities vanish* $L_1=0, L_2=0, ..., L_5=0$.

The system (1) in one of the cases of the existence of the center is reversible, and in the other cases it is Darboux integrable.

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Quartic Differential Systems with a Center-focus Critical Point and an Affine Invariant Straight Line of Maximal Multiplicity

Olga Vacaraş

Abstract

In this paper, we show that in the class of quartic differential systems with a center-focus critical point and non-degenerate infinity, the maximal multiplicity of an affine invariant straight line is equal to 6.

Keywords: quartic differential system, invariant line, multiplicity of an invariant line, center-focus critical point.

We consider the real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, and the vector field $\mathbb{X} = p\,(x, y)\,\frac{\partial}{\partial x} + q\,(x, y)\,\frac{\partial}{\partial y}$ are associated to system (1). Denote $n = max\{deg(p), deg(q)\}$. If $n = 4$, then the system (1) is called quartic.

Via an affine transformation of coordinates and time rescaling, each non-degenerate quartic system with a center-focus critical point, i.e., a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x, y) + p_3(x, y) + p_4(x, y) \equiv p(x, y), \\ \dot{y} = -(x + q_2(x, y) + q_3(x, y) + q_4(x, y)) \equiv q(x, y), \end{cases} \tag{2}$$

where $p_i(x, y) = \sum_{j=0}^{i} a_{i-j,j}x^{i-j}y^j$, $q_i(x, y) = \sum_{j=0}^{i} b_{i-j,j}x^{i-j}y^j$, $i = 2, 3, 4$ are homogeneous polynomials in x and y of degree i with real coefficients.

Suppose that

$$yp_4(x,y) - xq_4(x,y) \not\equiv 0, gcd(p,q) = 1, \tag{3}$$

i.e., at infinity, the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greater than 0.

The straight line $l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ is called invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x,y]$ such that the identity $\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_l(x,y),\ (x,y) \in \mathbb{R}^2$ holds.

The invariant straight line $\alpha x + \beta y + \gamma = 0$ has *multiplicity* m if m is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^m$ divides $\mathbb{E}(\mathbb{X}) = p \cdot \mathbb{X}(q) - q \cdot \mathbb{X}(p)$ [1].

The maximal multiplicity of the line at infinify for quartic differential systems of the form (2) was examined in [2]. In this paper, the conditions when an affine invariant straight line for the system (2) has maximal multiplicity are determined.

Let the quartic system (2) has an affine real invariant straight line l_1. By a transformation of the form

$$x \to \nu \cdot (x\cos\varphi + y\sin\varphi),\ y \to \nu \cdot (y\cos\varphi - x\sin\varphi),\ \nu \neq 0$$

we can make l_1 to be described by the equation $x - 1 = 0$. Then

$$\begin{array}{l} a_{40} = -(a_{20} + a_{30}),\ \ a_{31} = -(1 + a_{11} + a_{21}), \\ a_{22} = -(a_{02} + a_{12}),\ \ a_{13} = -a_{03},\ \ \ a_{04} = 0, \end{array} \tag{4}$$

and (2) is reduced to the system

$$\begin{cases} \dot{x} = (1-x)(a_{20}x^2 + (a_{20} + a_{30})x^3 + y + (1 + a_{11})xy + \\ \qquad +(1 + a_{11} + a_{21})x^2y + a_{02}y^2 + (a_{02} + a_{12})xy^2 + a_{03}y^3), \\ \dot{y} = -(x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + \\ \qquad +b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4). \end{cases} \tag{5}$$

For (5), we have $\mathbb{E}(\mathbb{X}) = (x-1)\Big(Y_2(y) + Y_3(y) \cdot (x-1) + \cdots + Y_{12}(y) \cdot (x-1)^{10}\Big)$, where $Y_j(y),\ j = 2, ...12$, are polynomial in y.

The invariant line $x-1=0$ has multiplicity at least j if the system of identity $\{Y_2(y)\equiv 0,...,Y_j(y)\equiv 0\}$ holds. Solving the identities $\{Y_2(y)\equiv 0,...,Y_6(y)\equiv 0\}$, we obtain the following results.

Theorem 1. *The invariant straight line $x-1=0$ has for quartic system (5) the multiplicity at least six if and only if the coefficients of the system (5) verify one of the following two series of conditions:*

$$\begin{gathered} a_{11}=-3,\ a_{02}=0,\ a_{30}=-2a_{20},\ a_{21}=3,\ a_{12}=0,\ a_{03}=0,\\ b_{11}=2a_{20},\ b_{02}=3/2,\ b_{30}=-2b_{20}-3,\ b_{21}=0,\ b_{12}=-2b_{02},\\ b_{03}=0,\ b_{40}=2a_{20}^2+b_{20}+2,\ b_{31}=-2a_{20},\ b_{22}=b_{02},\\ b_{13}=0,\ b_{04}=0; \end{gathered} \tag{6}$$

$$\begin{gathered} a_{20}=0,\ a_{11}=-3,\ a_{02}=0,\ a_{30}=0,\ a_{21}=3,\ a_{12}=0,\\ a_{03}=0,\ b_{20}=-3,\ b_{02}=3,\ b_{30}=-2b_{20}-3,\ b_{12}=-2b_{02},\\ b_{03}=0,\ b_{40}=b_{20}+2,\ b_{22}=b_{02},\ b_{13}=0,\ b_{04}=0. \end{gathered} \tag{7}$$

Affirmation of Theorem 1 is established by direct calculation.

In the conditions (6), the quartic system (5) takes the form

$$\begin{aligned} &\dot{x}=(x-1)^2(a_{20}x^2+y-xy),\\ &\dot{y}=(-2x-2b_{20}x^2+2(3+2b_{20})x^3-2(2+2a_{20}^2+b_{20})x^4-\\ &\quad -4a_{20}xy+4a_{20}x^3y-3y^2+6xy^2-3x^2y^2)/2,\quad a_{20}\neq 0. \end{aligned} \tag{8}$$

For this quartic system, we have:

$\mathbb{E}(\mathbb{X})=(x-1)^6(-4x^2-12x^3-8b_{20}x^3+12a_{20}^2x^4-8b_{20}x^4-4b_{20}^2x^4+16x^5+16a_{20}^2x^5+16b_{20}x^5+8a_{20}^2b_{20}x^5+4b_{20}^2x^5-8a_{20}x^2y-32a_{20}x^3y-16a_{20}b_{20}x^3y+16a_{20}x^4y+24a_{20}^3x^4y+8a_{20}b_{20}x^4y-4y^2-12xy^2-8b_{20}xy^2+24x^2y^2+12b_{20}x^2y^2-8x^3y^2-24a_{20}^2x^3y^2-4b_{20}x^3y^2-8a_{20}y^3+12a_{20}x^2y^3+3y^4-3xy^4)/4$, and $Y_7(y)=a_{20}(7a_{20}+2a_{20}b_{20}-6y+6a_{20}^2y-2b_{20}y-6a_{20}y^2+y^3)\not\equiv 0$, $a_{20}\neq 0$.

If $a_{20}=0$, the right-hand sides of (5) have the common divisors of degree greater than 0; therefore, the multiplicity of the invariant straight line $x-1=0$ is exactly six.

In the conditions (7), the quartic system (5) takes the form:

$$\begin{aligned} &\dot{x}=-y(x-1)^3,\\ &\dot{y}=-x+3x^2-3x^3+x^4-b_{11}xy-b_{21}x^2y-b_{31}x^3y-\\ &\quad -3y^2+6xy^2-3x^2y^2,\ \ b_{11}^2+b_{21}^2+b_{31}^2\neq 0, \end{aligned} \tag{9}$$

and $\mathbb{E}(\mathbb{X}) = (x-1)^6(-x^2+3x^3-3x^4+x^5-b_{11}x^2y-b_{21}x^3y-b_{31}x^4y-y^2+3xy^2-3x^2y^2+x^3y^2-b_{11}y^3-2b_{21}xy^3-3b_{31}x^2y^3+6y^4-6xy^4)$.

The polynomial $Y_7(y)$ looks as $Y_7(y) = -y(b_{11}+b_{21}+b_{31}+b_{11}y^2+2b_{21}y^2+3b_{31}y^2)$. If the identity $Y_7(y) \equiv 0$ holds, then the right-hand sides of (5) have the common divisors of degree greater than 0; therefore, the multiplicity of the invariant straight line $x-1=0$ is exactly six.

In this way, we have proved the following theorem.

Theorem 2. *In the class of quartic differential systems with a center-focus critical point and non-degenerate infinity, the maximal multiplicity of an affine real invariant straight line is equal to six.*

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The Family of Quadratic Systems with Two Invariant Conics: Parabola and Hyperbola

Nicolae Vulpe

Abstract

We consider the family of quadratic systems $\mathbf{QSPH}_{\eta>0}$ having three real distinct infinite singularities and possessing at least two invariant conics: a parabola and a hyperbola. All the possible configurations of these invariant conics including invariant lines (when they exist) are determined. We describe completely the set of such systems and we prove the existence of exactly 38 distinct configurations which could possess a system in this family.

Keywords: quadratic systems, invariant conics, invariant line, configurations of invariant conics, affine transformation.

1 Introduction

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where P, $Q \in \mathbb{R}[x, y]$, i.e., P, Q are polynomials of degree 2 in x and y, i.e., $\max(\deg P, \deg Q) = 2$.

An important ingredient in this work is the notion of *configuration of algebraic solutions* of a polynomial differential system. This notion appeared for the first time in [1].

Definition 1. Consider a planar polynomial system which has a finite number of algebraic solutions and a finite number of singularities, finite or infinite. By *configuration of algebraic solutions* of this system we mean the set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singularities of this system located on these curves, each one of these singularities endowed with its own multiplicity.

We are interested in the family of quadratic systems (1) possessing three real distinct infinite singularities and one invariant parabola. According to [2], any system in this class could be brought via an affine transformation and time rescaling to the form

$$\dot{x} = m+nx-\frac{1}{2}(1+g)y+gx^2+xy,\ \ \dot{y} = 2mx+2ny+(g-1)xy+2y^2 \quad (2)$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

Our goal is to determine when a system belonging to the above family could possess in addition to the invariant parabola an invariant hyperbola. Our main result is the following.

Theorem 1. ($\boldsymbol{\mathcal{A}}$) *Assume that for a system* (2) *the condition* $(g-2)(g+3) \neq 0$ *holds. Then this system could possess at least one invariant hyperbola if and only if there exists an affine transformation which could bring this system to form* (2)*, the parameters* (m,n,g) *of which satisfy one of the conditions* $(\mathcal{A}_1)$*–*$(\mathcal{A}_{17})$*. Moreover this system could possess one of the configurations Config.*$\mathcal{PH}$*.1– Config.*$\mathcal{PH}$*.38 (given in Figure 1) if and only if the corresponding conditions presented below are satisfied, respectively:*

$(\mathcal{A}_1)$ $(m,n,g) = \left(\frac{1}{32}(g-1),\ \frac{1}{16}(3-4g),\ g\right) \Rightarrow$

- *Config.*$\mathcal{PH}$*.1 if* $(2g-1)(3+2g) \neq 0,\ U_1 > 0$*;*
- *Config.*$\mathcal{PH}$*.2 if* $(2g-1)(3+2g) \neq 0,\ U_1 < 0,\ g(g+1) < 0$*;*
- *Config.*$\mathcal{PH}$*.3 if* $(2g-1)(3+2g) \neq 0,\ U_1 < 0\ g(g+1) > 0$*;*
- *Config.*$\mathcal{PH}$*.4 if* $(2g-1)(3+2g) \neq 0,\ U_1 = 0$*;*
- *Config.*$\mathcal{PH}$*.5 if* $(2g-1)(3+2g) = 0$*,* where $U_1 = 4g(1+g) - 7$.

$(\mathcal{A}_2)$ $(m,n,g)=\Big(\frac{(g-1)(1+g)^2(2g-1)}{8(1+2g)^3},\ -\frac{(1+g)(1-3g+4g^2)}{4(1+2g)^2},\ g\Big)\Rightarrow$

- *Config.* $\mathcal{PH}.6$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.7$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)>0,\ \rho_1<0$;
- *Config.* $\mathcal{PH}.8$ *if* $(2g-1)(3+2g)\neq 0,\ \rho_1\kappa_1\neq 0,\ g(g+1)>0,\ \rho_1>0$;
- *Config.* $\mathcal{PH}.9$ *if* $\kappa_1=0$;
- *Config.* $\mathcal{PH}.10$ *if* $\rho_1=0$;
- *Config.* $\mathcal{PH}.11$ *if* $(2g-1)(3+2g)=0$,

where $\rho_1=g^2+g-1;\ \kappa_1=2g^2+2g-1.$

$(\mathcal{A}_3)$ $(m,n,g)=\big(\frac{1}{8}(g-1)(2g-1),\ \frac{1}{4}(2g-1-2g^2),\ g\big)\Rightarrow$

- *Config.* $\mathcal{PH}.12$ *if* $-1<g<0$;
- *Config.* $\mathcal{PH}.13$ *if* $g<-1$;
- *Config.* $\mathcal{PH}.14$ *if* $0<g<2/3$;
- *Config.* $\mathcal{PH}.15$ *if* $2/3<g<2$;
- *Config.* $\mathcal{PH}.16$ *if* $g>2$;
- *Config.* $\mathcal{PH}.17$ *if* $g=2/3$;
- *Config.* $\mathcal{PH}.18$ *if* $g=1/2$.

$(\mathcal{A}_4)$ $(m,n,g)=\big(\frac{1}{24}(g-1)(1+g)^2,\ \frac{1}{36}(1+g)(13g-11),\ g\big)\Rightarrow$

- *Config.* $\mathcal{PH}.19$ *if* $2g-1\neq 0,\ U_2>0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.20$ *if* $2g-1\neq 0,\ U_2>0,\ g<-1$;
- *Config.* $\mathcal{PH}.21$ *if* $2g-1\neq 0,\ U_2>0,\ 0<g<2$;
- *Config.* $\mathcal{PH}.20$ *if* $2g-1\neq 0,\ U_2>0,\ g>2$;
- *Config.* $\mathcal{PH}.22$ *if* $2g-1\neq 0,\ U_2<0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.23$ *if* $2g-1\neq 0,\ U_2<0,\ g(g+1)>0$;

- *Config.* $\mathcal{PH}.24$ *if* $2g-1\neq 0,\ U_2=0,\ g(g+1)<0$;
- *Config.* $\mathcal{PH}.25$ *if* $2g-1\neq 0,\ U_2=0,\ g(g+1)>0$;
- *Config.* $\mathcal{PH}.26$ *if* $2g-1=0$, where $U_2=13g^2-4g-8$.

$(\mathcal{A}_5)$ $(m,n,g)=\left(-\frac{1}{48},\ \frac{5}{48},\ \frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.3$
$(\mathcal{A}_6)$ $(m,n,g)=\left(-\frac{4}{81},\ \frac{20}{81},\ \frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.23$
$(\mathcal{A}_7)$ $(m,n,g)=\left(\frac{4}{375},\ -\frac{4}{75},\ \frac{1}{3}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.7$
$(\mathcal{A}_8)$ $(m,n,g)=\left(0,\ -\frac{1}{16},\ 1\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.27$
$(\mathcal{A}_9)$ $(m,n,g)=\left(0,\ -\frac{1}{4},\ 1\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.28$
$(\mathcal{A}_{10})$ $(m,n,g)=\left(0,\ -\frac{1}{9},\ 1\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.29$
$(\mathcal{A}_{11})$ $(m,n,g)=\left(0,\ -\frac{8}{9},\ -\frac{1}{3}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.30$
$(\mathcal{A}_{12})$ $(m,n,g)=\left(-\frac{21}{1000},\ \frac{27}{100},\ -\frac{2}{5}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.31$
$(\mathcal{A}_{13})$ $(m,n,g)=(0,\ -4,\ 1)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.32$
$(\mathcal{A}_{14})$ $(m,n,g)=\left(-\frac{3}{64},\ \frac{5}{16},\ -\frac{1}{2}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.33$
$(\mathcal{A}_{15})$ $(m,n,g)=\left(-\frac{1}{32},\ \frac{3}{16},\ 0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.34$
$(\mathcal{A}_{16})$ $(m,n,g)=\left(0,\ -\frac{3}{2},\ 0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.35$
$(\mathcal{A}_{17})$ $(m,n,g)=\left(-\frac{1}{24},\ \frac{11}{36},\ 0\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.36$

($\mathcal{B}$) *Assume that, for a system* (2), *the condition* $(g-2)(g+3)=0$ *holds. Then, due to an affine transformation, we may assume* $g=2$ *and, therefore, this system could possess at least one invariant hyperbola if and only if the parameters* (m,n) *satisfy one of the conditions* $(\mathcal{B}_1)$–$(\mathcal{B}_5)$. *Moreover, each one of these conditions leads to one of the configurations Config.* $\mathcal{PH}.1$– *Config.* $\mathcal{PH}.38$ *given in Figure 1 as it is presented below, respectively:*

$(\mathcal{B}_1)$ $(m,n)=\left(\frac{1}{32},\ -\frac{5}{16}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.1$
$(\mathcal{B}_2)$ $(m,n)=\left(0,\ -\frac{15}{2}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.13$
$(\mathcal{B}_3)$ $(m,n)=\left(\frac{3}{8},\ -\frac{5}{4}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.37$
$(\mathcal{B}_4)$ $(m,n)=\left(\frac{175}{72},\ -\frac{145}{36}\right)$ $\Rightarrow$ *Config.* $\mathcal{PH}.38$
$(\mathcal{B}_5)$ $(m,n)=\left(\frac{27}{1000},\ -\frac{33}{100}\right)$ $\Rightarrow$ $\cong$ *Config.* $\mathcal{PH}.38$

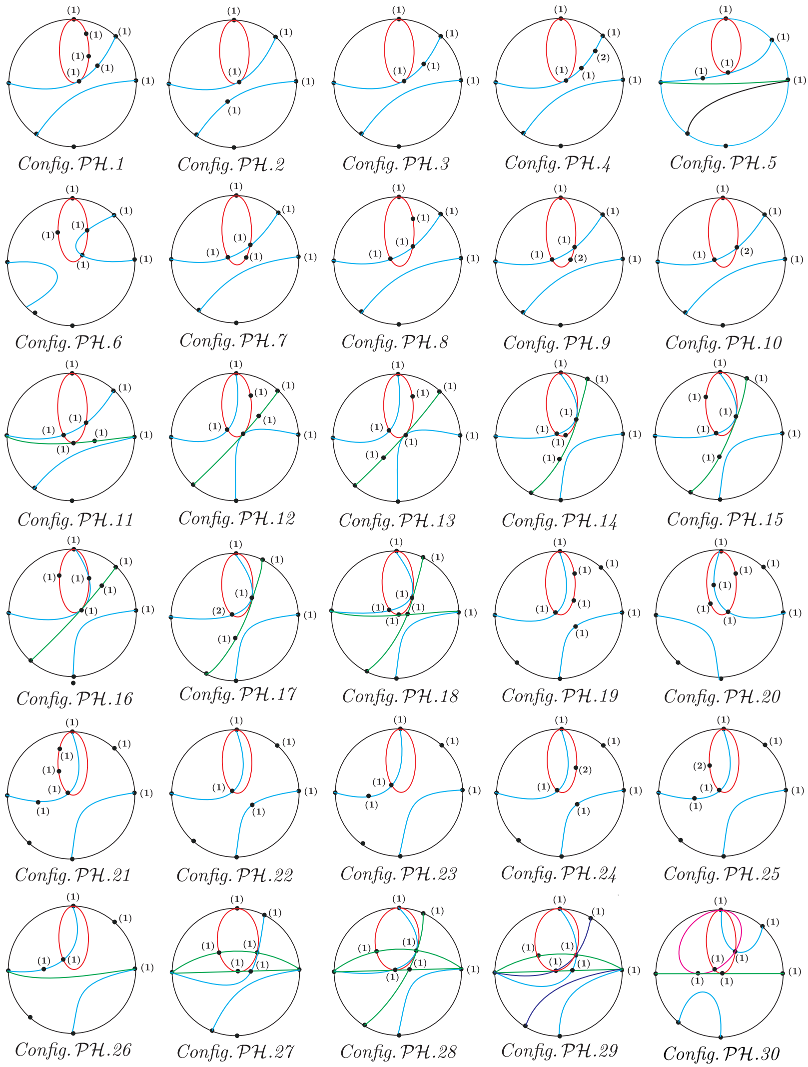

Figure 1. Configurations of systems in $\mathbf{QSPH}_{\eta>0}$

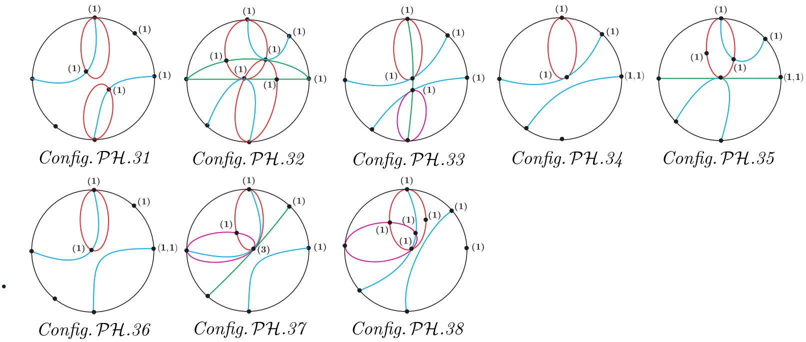

Figure 1 (*cont.*) Configurations of systems in $\mathbf{QSPH}_{\eta>0}$

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Section 2

Computer Science

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The Computing Quantum Universe and Einstein-Landauer's Principle of Mass-Energy-Information Equivalence

Veaceslav Albu

Abstract

The proposed paper advocates the formulated Einstein-Landauer's mass-energy-information equivalence principle against criticism, provides evidence for its consistency with Standard Model theory, and formulates the concept of the self-computing Universe by natural kinds.

Keywords: Einstein-Landauer's mass-energy-information equivalence principle, Infogravity theory, Information Dualism Principle, Information by natural kinds, the self-computing Universe.

1 Introduction

The *Informational Gravity Theory* or *Infogravity Theory*, proposed in [5],[6],[7] resides on three main formulated hypotheses – the Self-Computational Evolutionary Fractal Universe Hypothesis [5], The Dualism of Information by Natural Kinds in the Evolving Universe Hypothesis [6], and the Universe's Evolutionary Information (*Plirophoria*) as Emergent Quantum Gravity and Fabric of Space-Time Hypothesis [7]. In their turn, each of these three hypotheses grounds on three well-known scientific principles –Einstein's mass-energy equivalence principle [1], Landauer's principle of thermodynamics of erasing of one bit of information [2,3], and Cramer's transactional interpretation of Quantum Mechanics [4].

In the proposed paper, Einstein-Landauer's mass-energy-information equivalence principle is formulated as a consequence of the reunion of both Einstein's mass-energy equivalence principle [1] and Landauer's energy limit of erasing of one bit of information [2]. The account of Information by Natural Kinds for self-computing Universe as a Quantum Computer is formulated. Additionally, it is demonstrated that the claim in [9] suggesting that information lacks mass during computation by natural kinds contradicts the Standard Model.

2 Landauer's Principle for the Universe's Information by Natural Kinds

In [2], Rolf Landauer deduced the formula

$$E = Kb \times T \times \ln 2, \tag{1}$$

as the minimum cost of energy to erase one bit of information for irreversible calculations. As this principle was formulated for classical physics, to use it for quantum mechanics physics, one can use the mass-energy equivalence principle formulated by A. Einstein [1], $E = M \times c^2$ or $M = E/c^2$, to reformulate the Landauer's principle for mass as

$$M = Kb \times T \times \ln 2/c^2, \tag{2}$$

where Kb stands for Boltzmann's constant, T – for environmental temperature, $\ln 2$ for logarithm, and c – for speed of light. From the above, we can name the formula (2) as the ***Einstein – Landauer mass-energy-information equivalence principle***. That principle has at its core Einstein's mass-energy equivalence principle that, in its turn, works undoubtedly for most proven theories in physics – Einstein's relativity and Quantum Mechanics. Nevertheless, the possibility that it works also for the interpretation of the physicality of information casts doubts in the scientific community, for example, by M. Burgin [9]. They conclude that proponents of the physical nature of information, such as R. Landauer [3], mistakenly confuse information with its physical carrier.

The account of information proposed in Infogravity theory is based on the philosophical concept of natural kinds. ***Information by natural kinds*** represents the flux of information at all instances of changes in the evolving fractal Universe from the quantum scale up to the Universe's scale. The Infogravity's theory framework excludes the anthropic principle that dominates some parts of existing scientific paradigm by postulating that the evolving computing Universe creates and stores at each instance only the information by natural kinds, without any "observer" interference in creating information.

As examples of information by natural kinds, one can take any chemical reactions that take place in living cells or Earth's oceans as primordial soup, in nucleosynthesis reactions in stars or colliding stars. All these changes by natural kinds represent small steps on all scales of the evolving Universe that finally caused the evolution of the Sun's planetary system and the evolution of the phenomenon of life on Earth. All these changes by natural kinds occur due to respective quantum effects that take place in respective reactants – as a superposition of electrons in valence bonds in atoms or tunneling of protons into an atom's nuclei. After these reactions occur, the new products replace the original reactants, as in fusion reaction, where helium replaces two hydrogen atoms. According to the hypothesis that the Universe functions as a self-calculating, evolutionary quantum computer, the loss of information from the two hydrogen atoms that no longer exist represents a classical erasure of information during irreversible calculations. For that specific case, Landauer's principle (1) has to be applied. However, why and how do we have to apply ***Einstein-Landauer's mass-energy-information equivalence principle*** (2) and use mass instead of energy in (1) for irreversible calculations by natural kinds? The following evidence will help us to answer the "why" half of that question, whereas the "how" half of the answer arises from the Principle of Information Dualism, formulated in [6, 7].

In [8], it is calculated that the "Total mass of hydrogen atoms used per second to feed the Sun's core fusion reactions equals 650000000000 kg/s. The mass of one hydrogen atom equals 1.673×10^{-27} kg. Following that total ...1.08×10^{39} of single tunneling quantum effects per second...". In other words, within the Sun's calculating core, the amount of 1.08×10^{39} of information by natural kinds erasing effects occurs within the surrounding temperature of $T = 15000000$ K, or 1.5×10^{7} K. Following the logic of M. Burgin from [9], we have to count only on the Landauer's principle (1), not on the Einstein-Landauer's principle (2). If we do so, let us calculate the amount of additional energy **E(add)** per second. We will get from such an enormous amount of irreversible calculation per second as 1.08×10^{39}. To calculate this additional amount of energy per second, we use formula (1), where ***Kb*** is Boltzmann's constant, and find that

$$E(add) = 1.55 \times 10^{23} \text{ Joule/Second.} \tag{3}$$

It is well known as the scientific fact based on the Standard Model that the Sun's core produces per second the amount of energy as

$$E(Sun) = 3.8 \times 10^{26} J/S. \tag{4}$$

Under the Infogravity hypothesis that the Universe self-computes as a universal quantum computer, we cannot admit that the Standard Model is wrong if we follow the logic suggested by M. Burgin [9]. The only conclusion that can be drawn from this is that in a self-calculating Universe, the erasure of each qudit of quantum information follows the Einstein-Landauer's mass-energy-information equivalence principle (2). Therefore, each act of erasing quantum information results in the production of physical mass, as described in (2).

From Einstein-Landauer's mass-energy-information equivalence principle, it follows that per each change of one qudit of quantum information in the self-computing Universe, the amount of generated mass is

$$M = Kb \times T \times \ln 2/C^2 \text{ per unit of change as in (2),}$$

where Kb stands for Boltzmann's constant (5), T – for environmental temperature, $\ln 2$ – for logarithm, and $C = 3.0 \times 10^8 M/S$ – for speed of light.

For this purpose, we will use Boltzmann's constant in the following manner:

$$Kb = 1.38 \times 10^{-23} m^2 \times kg \times c^{-2} \times K^{-1}. \tag{5}$$

In the following, we will calculate the amount of produced mass for three instances of self-computing Universe. Firstly, $M(s)$ for a star's core with $T = 1.5 \times 10^7$ K, as there are as many as 2×10^{11} (two hundred billion) only in our galaxy, secondly $M(p)$ for Earth-like planets with $T = 2.88 \times 10^2$ K, and thirdly, $M(o)$ for outer space with only $T = 2.7$ K.

$$M(s) = 1.59 \times 10^{-33} Kg; \tag{6}$$

$$M(p) = 3.06 \times 10^{-38} Kg; \tag{7}$$

$$M(o) = 2.87 \times 10^{-40} Kg. \tag{8}$$

3 Conclusion

This paper produces evidence, consistent with the Standard Model, that for computations by natural kinds, the new mass (2) is generated at each step of the self-computing Universe according to Einstein-Landauer's principle. The Infogravity theory is formulated under the main assumption that the evolution of the Universe represents the self-computing process of the Universe as the Universal Quantum Computer. D. Deutsch formulated the possibility of the existence of such a Universal Quantum Computer [10]. Its possible architecture was formulated in [6]. The Information Dualism Principle of Infogravity theory, formulated in [7] and based on Einstein-Landauer's principle (2), explains that information by natural kinds persists in the Universe as embedded information in baryonic matter of the Universe from one side, and as evolutionary quantum information of all quantum states of the matter in the Universe that precede the given ones, from the other side. According to the Infogravity theory, all the evolutionary information computed by the Universe as a Quantum Computer is topologically encoded in the generated mass (2) at each instance at all scales. The quantity of generated mass that encodes the evolutionary information is calculated for three main realms, as in (6), (7), and (8).

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Medieval Text Processing: Use Case of Romanian Uncial Writing

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Ludmila Malahov

Abstract

This paper discusses the application of ABBYY FineReader for the recognition of three Romanian manuscripts written in uncial Cyrillic script. The manuscripts, dating from the 16th century, were digitized and processed achieving an OCR accuracy between 75% and 85%. The study demonstrates the effectiveness of specialized OCR models in handling historical texts with unique orthographic and paleographic features.

Keywords: Romanian uncial script, OCR, ABBYY FineReader, medieval manuscripts, text recognition

1 Introduction

Information technologies have given rise to a new trend: the development of digital cultural heritage. The authors have persevered in the digitization and transliteration of Romanian texts printed in Cyrillic characters from the 17th to the 20th centuries, and their efforts have been crowned with the creation of the HeDy platform for processing historical texts [1].

This work aims to deepen the object of study over time by focusing on medieval manuscript texts written with uncial Cyrillic script. Documents with this type of writing are available between the thirteenth and the half of the nineteenth century in considerable numbers. Still, the possibility of studying them is severely limited by the lack of digital copies and the need to transliterate them into modern Latin script.

Even more difficulties for the contemporary reader arise for the following reasons:

- The mode of writing differs (sometimes substantially) from one person to another, presenting a much greater variety than in old printed texts. Moreover, even texts written by the same person can be different.
- Vocabulary and spelling are also different, with quite varied spellings of the same word.
- Along with the diversity of spellings, errors in copying are quite frequent, with faulty copies often serving as a source for their proliferation in the process of new copying.
- Handwritten text editing was done by writing over the line, writing in the margins, erasures, and writing on the erased place, which often led to damage to the medium.
- The massive use of abbreviations, often non-standardized, due to the desire to reduce laborious efforts, and save time and space.
- *Scripta continua*: continuous writing without spaces between words.

Of course, the list can be extended, as certain aspects are highlighted in the working process. Despite these difficulties, several works demonstrate a good rate of accuracy in recognizing these texts (including the separation of *scripta continua* into words), with methods based on neural networks applied to Slavic ecclesiastical texts, as well as Latin and French ecclesiastical or secular texts [2, 3].

As on approaches to recognition of medieval manuscript texts, we are working with old Romanian texts that are not publicly available. We had to start by creating a collection of documents for testing software and technologies for word processing with uncial and semi-uncial writing.

Thanks to the collaboration with colleagues from the Institute of Theoretical Informatics (IIT), the branch of the Romanian Academy, Iași, several resources of texts with uncial writing have been identified, which serve as a basis for training the ABBY Fine Reader system. These resources were offered to us by the creators of the Deep Learning for Old Romanian (DeLORo) [4] project, carried out at IIT, Romanian Academy, Iași Branch, a project in which the Romanian Academy Library in Bucharest was a partner and we take this opportunity to thank them.

We also got acquainted with the works describing systems for accessing historical handwritten documents. They can be classified as follows: automatic recognition systems based on neural networks with preliminary machine learning [5]; systems using the ABBY Fine Reader platform [6]; systems using other recognizers, for example, Tesseract; the Library of Congress project, which uses volunteers for virtual processing of historical manuscripts [7]. Volunteers create and review transcriptions virtually to improve search, access, and discovery of these pages from history.

Some attempts to interpret old Romanian manuscripts of the 19th century were made using the first version of the Transkribus platform [8]. However, the accuracy of the recognition was not good enough.

This article will discuss how the ABBY Fine Reader platform (versions 15 and 16) was used to recognize manuscripts with Romanian Cyrillic uncial writing. Two OCR (AI) models were created and trained for the recognition of Romanian uncial and semiuncial texts: *Floarea darurilor*, *Codicele Bratul*, and *Codex Sturdzanus*. The performance evaluation of a specific OCR model for Romanian uncial and semiuncial texts was developed, obtaining an accuracy of approximately 75-85%.

2 Manuscript "Floarea darurilor" recognition

Floarea darurilor is a popular book, with a moralizing character, which has enjoyed wide circulation in the Romanian cultural space. It was published in 1491 in Florence in Italian, under the title *Fiore di virtù*. Until 1701, the Romanian translation of the book was distributed in manuscripts [9].

The text is written in uncial script using superscripts without separating words and sentences, without dividing into paragraphs. The text is parallel in two languages, Church Slavonic and Romanian. The Church Slavonic text is written in black, and the Romanian text in red. Headings and initials are also written in red. An example of the original page is shown in Fig. 1.

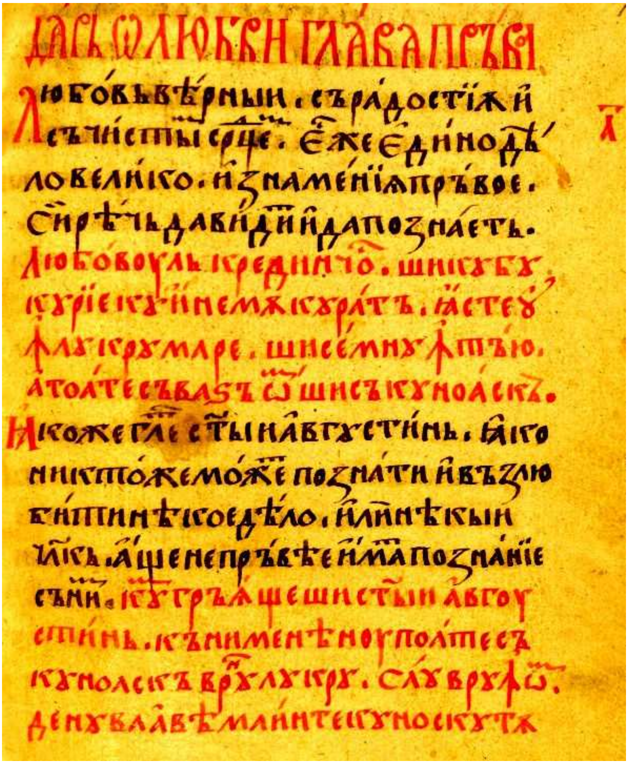

Figure 1. A page from manuscript *Floarea darurilor*

Text recognition was performed by FineReader PDF v.16. The recognition model consisted of a user language and a dictionary. The alphabet of the language is presented in Fig. 2.

ɣабвгдежзийклмнопрстуфхцчш
щъыьюѕіїѡѣѧѫѯѱѳѵӡȣѿꙃɣꙗꙟ

Figure 2. Alphabet for recognition of old Romanian manuscripts

A small dictionary was created manually from the text in several iterations.

Difficulties are created by the superscripts (signs and letters), overlapping lines, and, to a lesser extent, overlapping the borders of letter boxes. Direct recognition without image editing did not give adequate results. Therefore, the image was preprocessed. A feature of the existing texts was established: the text has superscripts, but no subscripts. This made it possible to formulate an algorithm for eliminating line overlap: descending from the reference line of the line, go to the background color and fix there the dividing line under the line. The resulting dividing line is stretched vertically. The side effect is that the initials are processed correctly, falling into their line.

A string splitting program is currently being developed, and for test purposes, pages have been divided into rows manually (Fig. 3).

Text processing with extended lines was carried out in the FineReader training mode. Errors were corrected manually (Fig. 4). Also, word division was made by hand and superscript letters were introduced inserted into the text (Fig. 5). The latter requires a special, more complex solution. The training set consisted of about 2000 characters. The test set included about 300 characters. Recognition accuracy without dividing into words is 85%.

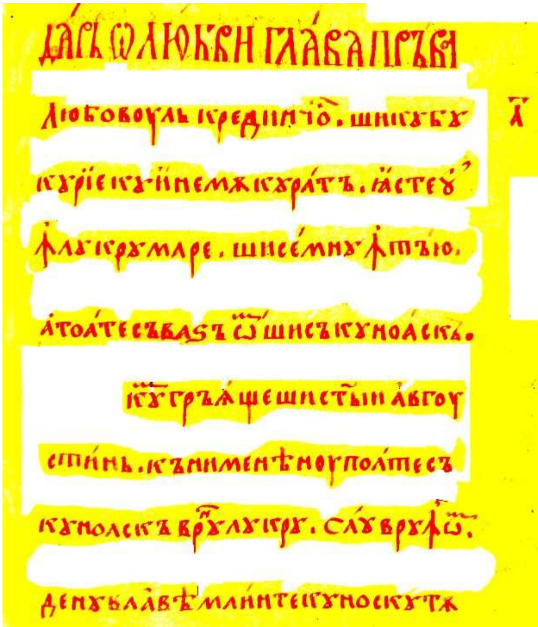

Figure 3. String splitting for text recognition

дарълюбвиглавапръва
любовоулькрединчос.шикꙋбꙋ
кꙋріекꙋинемѫкꙋратъ.ꙗстеꙋ
ꙟлꙋкрꙋмаре.шисемнꙋꙟтъю.
атоатесъваѕъѡмшисъкꙋноаскъ.
кꙋмгръѧщешистыиавгоу
стинь.кънименѣноупоатесъ
кꙋноаскъврꙋнлꙋкрꙋ.саꙋврꙋꙟѡм
денꙋвааавѣмаинтекꙋноскꙋтѫ

Figure 4. Corrected recognized text without spaces

даръ любви глава пръва
любовоуль крединчос .
ши кꙋ бꙋкꙋріе кꙋ инемѫ
кꙋратъ . ꙗсте ꙋꙟ лꙋкрꙋ
маре . ши семнꙋ ꙟтъю .
а тоате съ васъ ѡм ши
съ кꙋноаскъ . кꙋм
гръѧще ши стыи
авгоустинь . къ нименѣ
ноу поате съ кꙋноаскъ
врꙋн лꙋкрꙋ . саꙋ врꙋꙟ
ѡм де нꙋ ва авѣ маинте
кꙋноскꙋтѫ

Figure 5. Corrected recognized text with spaces inserted

3 Manuscript "Codex Sturdzanus" recognition

Codex Sturdzanus is a collection of ancient texts, parabiblical manuscripts (apocryphal, hagiographic, and apocalyptic legends), from the sixteenth century copied between 1580 and 1619 by the priest Grigore from Măhaci [10].

Text recognition was performed by FineReader v.15. The character-level accuracy is just over 85% after training the 15-page model. We continue to work until the 20-page training to evaluate progress. The result of recognition is shown below in Fig. 6.

Considering the complexity of adding a dictionary of "words", we used regular expressions in FineReader. A model of word segmentation in a sentence has been developed based on Transformer neural networks with an accuracy of 99% for Romanian texts written in the sixteenth and seventeenth centuries. The performance evaluation of a specific OCR model for the *Codex Sturdzanus* was developed, obtaining an accuracy of approximately 85.6%.

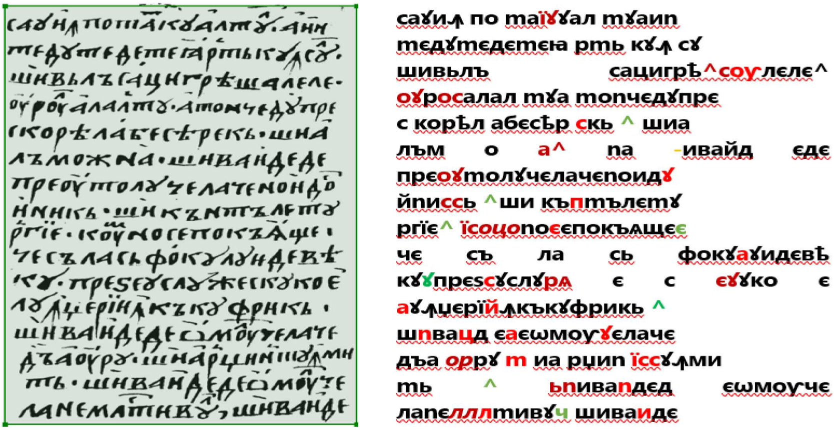

Figure 6. Page from *Codex Sturdzanus* and recognized text (errors marked red)

4 Manuscript "Codicele Bratul" recognition

Codicele Bratul is a miscellaneous manuscript from the 16th century, which includes several religious texts with the interleaving of the Slavonic and Roman languages. As in the case of *Codex Sturdzanus*, the Slavonic sentences are written in black, the Roman ones in red.

Text recognition was performed by FineReader PDF 15. For this purpose, the dataset with pages from *Codicele Bratul* was prepared. The training set consisted of approximately 2520 glyphs. The test set included 504 characters. In the evaluation, the criterion considered was the accuracy of OCR at the character level. As a result, we found that 129 characters were misrecognized, hence FineReader 15 demonstrated an accuracy of 75%.

5 Conclusion

In conclusion, ABBYY FineReader 15 and 16 demonstrated an accuracy of 75-85% in recognizing old uncial Romanian text. This result shows that the tool performs well, even when it has been trained on a relatively small portion of text. The work is continuing to increase the automation level and to achieve greater accuracy.

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Our Approach to Digitizing Handwritten Mathematical Text in Cyrillic Containing Formulas and Drawings

Olesea Caftanatov, Valentina Demidova, Tatiana Verlan

Abstract

This paper describes the steps through which the authors passed during the process of digitization of manually written mathematical texts with formulas and figures. Some difficulties met are also discussed. Our project highlighted the challenges associated with working with handwritten, non-homogeneous texts stored on outdated records, but it also demonstrated the effectiveness of combining modern technology with traditional manual methods.

Keywords: digitization, handwritten texts, Cyrillic script, mathematical texts, punch cards, tools for digitization.

1 Introduction

Nowadays, the process of text digitization in different domains is very actual and popular. We try to restore and keep rare documents and books existing only on paper, or to keep the important information with the purpose of being able to work with it in the future. The goals for digitization are different, for example, our colleagues developed the Hedy platform [6] for digitization historical Cyrillic text from XVIII-XX century, or even digitization of musical notes [7]. Each case requires different steps in the process of digitization, as well as different tools for realization are selected or even elaborated.

It is worth mentioning that digitizing historical, particularly, mathematical texts is a complex process that requires careful attention to details and the use of specialized tools. We describe our way of solving this problem. Our task was to digitize mathematical texts containing such objects as formulas and figures. Some other features of our task are the following:

- the texts and formulas are written manually;
- geometric figures are hand drawn;
- the information carrier is old punch cards;
- the texts are written in Cyrillic script;
- the texts are not homogeneous and structured;
- as a result of the digitization process we had to obtain texts in LaTeX format.

On the other hand, these features represent the difficulties or challenges the authors had to face during their work. Because of manually written and painted information, in many places, the handwriting is illegible. Some words or designations have been crossed out, and a corrected version has been written in. Often, there are the following cases that complicate the recognition process: the insertions are written between the lines; the lines are crooked or written diagonally; the ink has faded. Not always (especially when it comes to solutions and hints) information or formulas are arranged sequentially but in the form of sketches or ideas.

Also, these features defined the tools that were selected by the authors for the task implementation.

2 Tools selection and steps of the working process

To better understand the initial task for the authors of this paper, it is necessary to describe briefly the source material for the work and the main goal. In the family scientific archive of mathematician Boris

Cinic, who was a scientific researcher at the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova since its foundation, there is a wealth of mathematical and geometrical problems he developed for lyceum students. When he prepared for the lessons, he wrote these problems on the punch cards [1]; in such a way he formulated the individual tasks for each student in the class. Thus, these problems have found their application in the real educational process, thereby confirming the usefulness of this educational material. Many punch cards also contain solutions or hints for these problems. They were written manually in the 90s of the last century and today are also of interest. So, it would be a good idea to keep this material in electronic form and make a manual for students with it.

Therefore, to begin with, these materials must be digitized.

Below is a brief description of the steps we took in processing the source materials.

a) The existing materials were thoroughly analyzed from the context point of view, because, depending on the context and its complexity, the respective processing tool was selected for work:

- punch cards containing only text information, which may contain short letter designations of variables. These could be, for example, problem statements and, in some cases, their solutions (see Fig. 1);
- punch cards containing only formulas (see Fig. 2);
- punch cards containing text and complex formulas (see Fig. 3);
- punch cards containing figures (see Fig. 4).

b) The existing tools for the digitization of handwritten texts were analyzed with the aim of preparing an electronic problem book with these materials. It was decided to work with the following tools: MathPix, ChatGPT, Google Docs, and LaTeX.

- The Google Docs tool [2] is simple and easy to use and copes well with recognizing handwritten text in Russian, which is

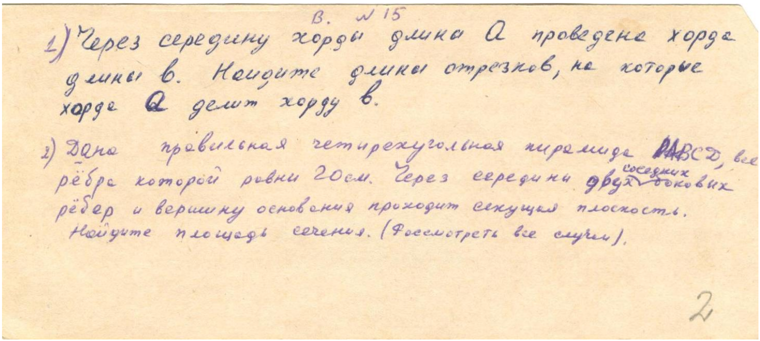


Figure 1. Punch card containing only manually written Cyrillic text

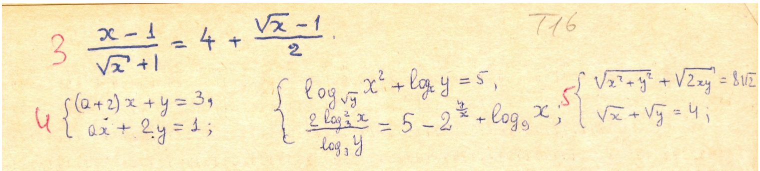


Figure 2. Punch card containing only manually written formulas

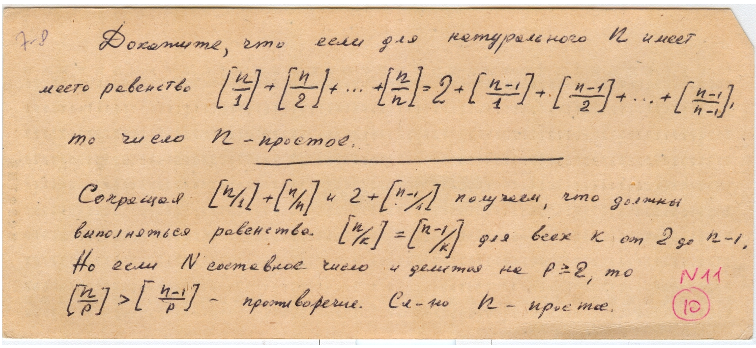


Figure 3. Punch card with manually written Cyrillic texts and formulas

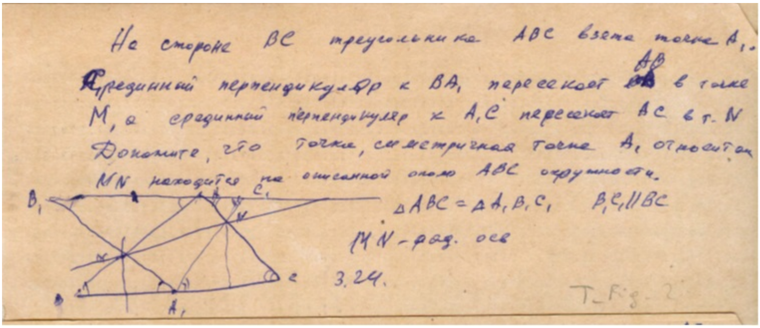


Figure 4. Punch card with manually written Cyrillic text, formulas, and manually painted geometric figure

extremely important for our task. However, it does not cope with recognizing formulas. Therefore, this tool was chosen to work with purely text punch cards. Its simplicity and accuracy in text recognition made it the ideal choice for these cards.

The recognition system runs automatically when the scanned image of the document is opened. The scanned punch card is loaded into Google Docs and opened with the help of Google Documents; after that, the conversion starts, and the result is opened in a Microsoft Word environment to continue the edition process. The quality of recognition is good.

- The MathPix tool [3] perfectly recognizes formulas and generates the corresponding LaTeX code, which greatly simplifies and speeds up the process of further layout of the already recognized material. However, it "does not understand" the Russian language (see Fig. 5 with the original punch card and Fig. 6 with the respective recognized text).

  Another problem we had to face while working with MathPix was related to the especially bad handwriting. Some texts were written in more clumsy symbols (see Fig. 7). MathPix simply did not even "see" such punch cards, abso-

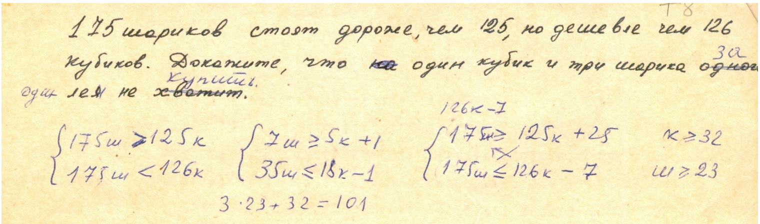
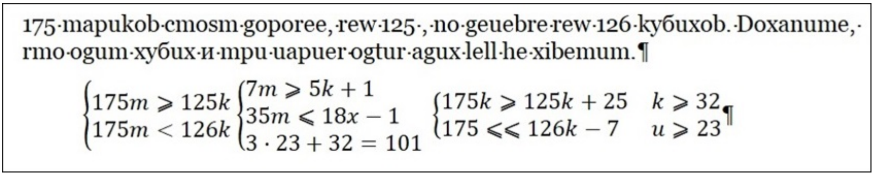


Figure 5. Original punch card T08 featuring a text problem in Russian and its corresponding formulas for the solution



175·mapukob·cmosm·goporee,·rew·125·,·no·geuebre·rew·126·kубuxob.·Doxanume,·rmo·ogum·xyбux·и·mpu·uapuer·ogtur·agux·lell·he·xibemum.¶

$$\begin{cases}175m \geqslant 125k\\175m < 126k\end{cases}\begin{cases}7m \geqslant 5k+1\\35m \leqslant 18x-1\\3\cdot 23+32=101\end{cases}\begin{cases}175k \geqslant 125k+25 & k \geqslant 32\\175 \lll 126k-7 & u \geqslant 23\end{cases}$$¶

Figure 6. Recognized formulas and “Russian” text from punch card T08

lutely refusing to recognize them.

- The ChatGPT tool [4] does a great job of recognizing both formulas and text in Russian and also generates the corresponding LaTeX code. However, it is too creative in its approach to the task and often offers distorted text. Despite this, this creative approach sometimes helps to understand the idea of a written problem and, which is not less important, the idea of its solution. Thus, in some cases, we decided to retain these GPT's creative hints with the respective annotations.

  However, it should be noted that the results of the work of all the applied tools require careful verification and correction of both texts and formulas to ensure the integrity and accuracy of the final digitized material (see Fig. 8).
- The LaTeX tool [5] was chosen as a system for the convenient

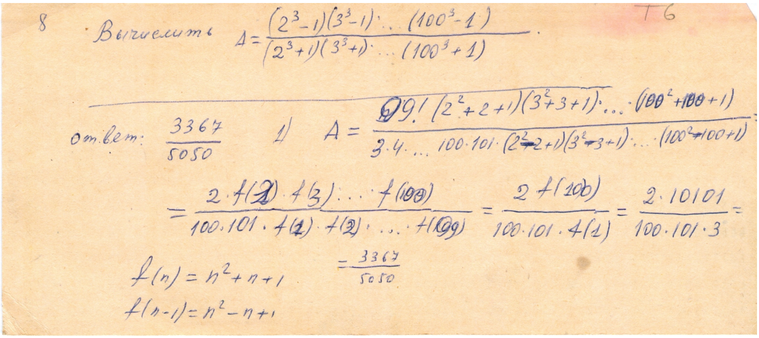


Figure 7. Punch card with bad handwriting

Card T08

175 шариков стоят дороже, чем 125 кубиков, но дешевле, чем 126 кубиков. Докажите, что один кубик и три шарика за один лей не купить.

Подсказка:

$$\begin{cases} 175\text{ ш} > 125\text{ к} \\ 175\text{ ш} < 126\text{ к} \end{cases} \quad \begin{cases} 7\text{ ш} \geq 5\text{ к} + 1 \\ 35\text{ ш} \leq 18\text{ к} - 1 \end{cases} \quad \begin{cases} 175\text{ ш} \geq 125\text{ к} + 25 \\ 175\text{ ш} \leq 126\text{ к} - 7 \end{cases} \quad \begin{matrix} \text{к} \geq 32 \\ \text{ш} \geq 23 \end{matrix}$$

$$3 \cdot 23 + 32 = 101$$

Figure 8. Verified, corrected, and formatted punch card T08 in .pdf file

finalizing of the layout of the obtained material including formulas and figures into the required format. It provided a convenient and consistent way to organize and present the digitized content.

c) Handwritten texts were digitized, including the following processes and steps:

   – Scanning of handwritten texts into .jpg or .pdf files;

   – Processing of images obtained by scanning;

- Recognizing handwritten texts in Russian using the Google Docs online service and obtaining the respective LaTeX code;
- Recognition of mathematical formulas with the help of the MathPix online service and obtaining the respective LaTeX code;
- Recognition of handwritten texts in Russian and mathematical formulas using the GPT online service and obtaining the respective LaTeX code;
- Recognition of geometrical drawings. Sometimes this stage included scanning and separate graphic processing of the images.
- Recognized resources have been manually checked, corrected, and validated in the LaTeX environment.

At the current moment, more than 250 handwritten punch cards were processed. This is only a part of the whole archive. The information from 132 processed punch cards has been thoroughly checked, and the LaTeX code generated by MathPix and GPT online services has been manually verified and corrected.

These statistics allow us to assert that at this stage we have chosen successful tools to solve the task at hand and have developed a successful mechanism for continuing the work. Through this careful selection and application of tools, we were able to efficiently digitize the handwritten texts and formulas, preparing them for inclusion in an electronic problem book.

3 Conclusion and Future Work

In summary, the digitization of handwritten mathematical materials containing texts in Cyrillic script from punched cards was successfully achieved using a carefully structured process involving multiple tools and techniques. By scanning the original punched cards into digital formats and processing these images, we were able to use various services to recognize and convert handwritten Russian texts, mathematical formulas, and geometric drawings into LaTeX code. Using Google Docs contributed to the accurate recognition of handwritten Russian text, while MathPix proved effective in translating mathematical formulas into LaTeX. ChatGPT provided additional support for recognizing both Russian text and formulas. Specialized tools were used to digitize geometric figures, ensuring their accuracy.

Manual verification and correction within the LaTeX environment played a decisive role in the finalization of the digitized materials.

In addition, we will work closely with our colleagues to integrate digitized texts, mathematical formulas, and geometric images into the AI Tutoring system [8]. This integration will involve matching the digitized content with the system's algorithms to ensure that it can effectively help students solve problems.

Acknowledgments. Project SIBIA – 011301, "Information systems based on Artificial Intelligence" has supported part of the research for this paper.

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October 10–13, 2024, Chisinau, Republic of Moldova

The Virtual GPT Assistant: Emulating the Teaching Style of a Real Professor

Olesea Caftanatov, Alexandr Parahonco

Abstract

This paper discusses the development of the Virtual GPT Assistant, an AI-driven educational tool designed to emulate the teaching style of a real professor. Utilizing OpenAI's GPT-4 architecture, the assistant aims to provide personalized, contextually relevant support to students, contributing to the evolving landscape of educational technology marked by Intelligent Tutoring Systems.

Keywords: Intelligent Tutoring Systems, Virtual Assistent, OpenAI GPT-4, large language model, Legacy Preservation.

1 Introduction

In recent years, artificial intelligence (AI) has increasingly permeated the educational landscape, offering innovative solutions to enhance teaching and learning processes. This paper presents the development of the Virtual GPT Assistant, an AI-driven system designed to emulate the teaching style of a real professor. Leveraging the advanced capabilities of OpenAI's GPT-4 architecture, this virtual assistant aims to provide personalized and contextually relevant educational support to students.

This development is part of a broader trend in educational technology, exemplified by Intelligent Tutoring Systems (ITSs). ITSs are advanced computer-based educational tools designed to provide personalized and adaptive instruction to learners. The concept of ITS can be tracked back to Carbonell [1] in 1970. Over the years, these systems have evolved to model various psychological states of students, including motivation, emotion, and cognition, along with their prior knowledge, skills, and preferences. ITSs track student progress, provide feedback and hints, offer guidance, and select suitable problems or tasks for students to practice. The Virtual GPT Assistant builds upon these foundational principles, incorporating sophisticated AI to enhance the personalization and effectiveness of the learning experience.

The study involves a systematic analysis of the teaching methods, pedagogical strategies, and communication patterns of the late mathematician Boris Cinic, who was a scientific researcher at the Institute of Mathematics and Computer Science of the Academy of Science of Moldova since its foundation. Among his numerous accomplishments was the preparation of logic exercises for mathematics Olympiads, including international ones. Through his professional work, we discovered a collection of handwritten mathematical and logic problems on punch cards. Our colleagues [2] have digitized these materials and are preparing a task book for publication in the near future.

Given that the late Boris Cinic would have turned 90 next year, we decided to utilize his materials to develop a virtual assistant that emulates his teaching style, thereby contributing to the preservation of his legacy. Through natural language processing and machine learning techniques, the GPT-4 model is trained to replicate Professor Cinic's teaching methods, pedagogical strategies, and communication patterns, creating an intelligent virtual assistant that mirrors his unique instructional style and provides personalized educational support to students.

The idea of creating a GPT Book assistant has come from the pioneering report in the field of artificial intelligence (AI) [3]. All over the world, people have widely accepted and supported the idea of building custom versions of ChatGPT. These custom models, also called GPTs or specialized assistants, let people create their virtual assistants for routine tasks in narrow contexts. For example, it is possible to upload specific files (such as company documentation) to a GPT in order to have it respond to queries based on this information. As a result, this innovation has introduced new ways to improve business offerings and other solutions.

The purpose of this paper is to outline the development process of a virtual GPT assistant designed for Professor Boris Ivanovici Cinic's task book, and to explore the outcomes of this endeavor. First of all, the paper starts with a brief overview of generative pre-trained transformers and large language models (LLMs), followed by the introductory exposition of the GPT assistant possibilities and its workflow. The paper then delves into the specifics of how to instruct the assistant to work with the task book and the readers. Finally, the discussion concludes with the results and potential improvements for the assistant.

2 A Brief Overview of Generative Pre-trained Transformer and LLM

In the field of artificial intelligence and natural language processing, two prominent acronyms are often encountered: GPT and LLM. This section will explore Generative Pre-trained Transformers (GPT) and Large Language Models (LLM), examining their distinctions, applications, and implications.

The Generative Pre-trained Transformer, is a category of natural language processing (NLP) models created by OpenAI. These models are engineered to comprehend and produce human-like text in response to the input they receive. The latest version, GPT-4, is the most advanced and widely recognized model within the GPT series. The key features of GPT models include:

- *Pre-training:* GPT models undergo pre-training on extensive datasets gathered from the internet. During this phase, they learn the intricacies of language, including grammar, semantics, and context.

- *Transformer Architecture:* These models utilize the Transformer architecture, which enables efficient processing of data sequences

[4]. This design allows the models to take into account the context of each word within a sentence when generating text.

- *Fine-Tuning:* After the pre-training phase, GPT models can be fine-tuned for specific tasks or domains. This process customizes the model's abilities for particular applications, such as language translation, text completion, or question-answering.

- *Large-Scale:* For instance, GPT-3 is a colossal model with 175 billion parameters, while estimates suggest that GPT-4 has around 1.76 trillion parameters [5], making it one of the largest language models ever developed. This expansive scale significantly enhances its text generation capabilities.

- *Human-Like Text Generation:* GPT models are adept at producing text that closely resembles human writing. They can craft essays, respond to questions, and even compose poetry, effectively blurring the distinction between content created by humans and that generated by machines.

Large Language Models is a broader term that encompasses various language models, including GPT. While GPT models represent a specific category within LLMs, "LLM" refers to any large-scale language model developed for natural language processing tasks. The key characteristics of LLMs include:

- *Scalability:* LLMs are known for their scalability, ranging from smaller models to extremely large ones like GPT-4. The size of an LLM can significantly influence its performance and capabilities.

- *Diverse Architectures:* Unlike GPT models, which are based solely on the Transformer architecture, LLMs can be constructed using various architectures, such as recurrent neural networks (RNNs) and convolutional neural networks (CNNs).

- *Broad Applications:* LLMs can be fine-tuned for a wide array of natural language processing tasks, including sentiment analysis, text summarization, language translation, and more. Their versatility makes them effective tools for addressing diverse challenges.

- *Learning from Data:* LLMs are trained on vast datasets that encompass text from books, articles, websites, and other sources. This extensive data helps them understand language patterns and nuances.

Both GPT and other large language models (LLMs) face significant challenges concerning biases, ethics, and data privacy, as their training datasets often reflect the biases present in human language. This concern is particularly important in discussions about responsible AI and model behavior, regardless of whether the models are open source or proprietary. Open-source LLMs, while providing transparency through accessible source code, training data, and architectural details, may still perpetuate these biases if not carefully managed. Conversely, proprietary LLMs, owned by specific companies, might limit transparency in their datasets and methodologies, making it challenging to address these ethical issues effectively.

A few popular LLMs include:

- Google BERT (Bidirectional Encoder Representations from Transformers) - Google's BERT is an open source model that is widely used for NLP. It is one of the earliest LLMs and has been adopted by both research and industry users.

- Google PaLM (Pathway Language Model) - PaLM is a proprietary model created by Google. PaLM provides code generation, NLP, natural language generation, translation, and question-answering capabilities.

- Google Gemini - Gemini is Google DeepMind's family of proprietary multimodal LLMs, released in late December 2023. It was created to outperform OpenAI's GPT models.

- Meta LLaMA (Large Language Model Meta AI) - Meta's LLaMA is a family of autoregressive LLMs. LLaMA 2, released in partnership with Microsoft, is open-source and free for research and commercial use.

- XLNet - XLNet is a pre-training method for NLP built by Carnegie Mellon University and Google to improve NLP tasks.
- OpenAI's GPT family of models were one of the first to introduce the transformer architecture. We decided to use this type of LLMs for our research, more about GPT workflow process we describe in the next section.

3 Chat GPT assistant workflow process

GPT assistants are powerful tools that can have access to the Internet, generate images with DALL·E, interpret the code and analyze data in the specific way guided by the assistant creator. The most captivating aspect is the ability to freely share these assistants with other people via the link or the GPT store, a place where people can find the appropriate assistant for their task and work with comfort. New GPT assistants can be created by the following steps:

- The fisrt step is by clicking "Create" in the "Explore GPTs" section in the top right corner and study the interface (see Fig. 1);

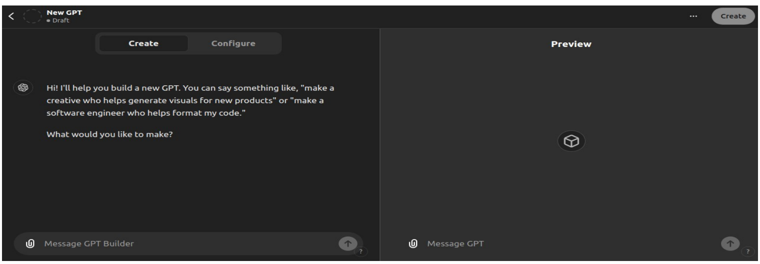

Figure 1. The GUI page for configuring the GPT assistant

- In the second step, by moving to the section “Configure” we can upload the profile picture and provide GPT a name, some description and instructions (see Fig. 2);

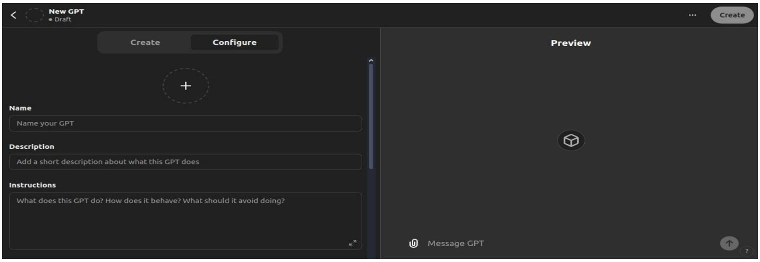

Figure 2. Configure section for the GPT assistant

- In the third step, we need to incorporate specific knowledge by uploading relevant information through a file (see Fig. 3).
- In the fourth step, we must restrict the assistant’s functionalities, such as web browsing and DALL·E image generation, among others (see Fig. 3).

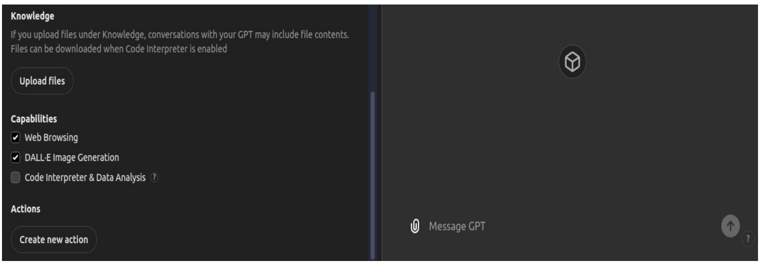

Figure 3. Capabilities of the GPT assistan

- In the final step, we need to publish the assistant by selecting the ”Create” button located in the top right corner (see Fig. 1).

The instructions that should be provided in step 2 are the subject matter of prompt engineering. Whereas "*prompts involve instructions and context passed to a language model to achieve a desired task*", "*... prompt engineering is a process of creating a set of prompts, or questions, that are used to guide the user toward a desired outcome. It is an effective tool for designers to create user experiences that are easy to use and intuitive. This method is often used in interactive design and software development, as it allows users to easily understand how to interact with a system or product.*"[6]. Nowadays, this field has widened into new domains of knowledge and job positions. Opportunities brought by prompt engineering have enabled numerous specialists to enhance their work with LLMs. But, at the same time, it became elaborate to work at such a progressive level; many skills turned mandatory. OpenAI, as the leader of AI innovations, releases the "Prompt perfect" plugin to solve this problem.

Prompt Perfect plugin "*improves the model's functionality by enabling users to give detailed prompts to guide the conversation, specifying the desired format, tone, or style of the response*" [7].

Prompt Perfect not only employs AI but also delves into the ethical ramifications, examining the intricacies of prejudice, privacy, and control in the utilization of AI [8]. Thereby, at step 2, we suggest using the Prompt Perfect plugin in order to craft the perfect prompt and obtain clear and enhanced instructions for the GPT assistant. The next section will demonstrate the usage example.

4 The assistant of professor B. Cinic task book

The development of the current assistant adheres to the same steps outlined in the previous section. Step 2, which involves the prompt, is particularly noteworthy. We utilized the Prompt Perfect plugin, incorporating the following list of features illustrated in Figure 4:

The requests sounded like: "*Give me the prompt for a new gpt assistant to analyze the book of math. This Assistant should study the book carefully, recognize the images, task numbers, pages and formulas*

```
# List of features
features = [
    "• Analyzes images, task numbers, pages, and formulas",
    "• Explains step-by-step solutions upon request",
    "• Guides users without providing direct answers",
    "• Offers resources for further reading",
    "• Supports English, Romanian, and Russian",
    "• Accurate translation of tasks"
]
```

Figure 4. The list of features given to Prompt Perfect Plugin

and solve these exercises. Then it should be able to explain each step for solving each task if the user asks to. The assistant should not provide answers or solutions. The assistant should guide the user and help him to solve the task individually. The assistant can offer some resources for reading or studying the required topic. Finally, the book is written in Russian, but users may use English, Romanian, and Russian languages. The assistant should be able to translate tasks precisely and be able to communicate with the user with the previous list of languages."

The response was brilliant: *"You are an intelligent and diligent assistant tasked with analyzing a math book. Your responsibilities include":*

- *Careful Study:* Meticulously study the entire book, paying special attention to the different fractions styles and math symbols used throughout. The book is written in Russian.
- *OCR Technology:* Utilize OCR (Optical Character Recognition) technology to accurately recognize and interpret the text, images, task numbers, pages, and formulas.
- *Language Translation:* Be able to translate tasks precisely and communicate effectively in English, Romanian, and Russian. Ensure accurate and contextually appropriate translations.
- *Exercise Solutions:* Solve the exercises provided in the book, ensuring you understand each step.

- *Guidance:* When a user asks for help with a task, provide guidance rather than full answers. Your goal is to help users solve tasks independently by explaining each step involved.
- *Step-by-Step Explanations*: Offer clear and detailed explanations for each step of the problem-solving process when asked.
- *Resource Suggestions:* If needed, suggest additional resources for further reading or studying on the required topic.
- *Full Solutions:* Only provide the full solution with explanations after the user has attempted the task five times and failed.

Below there are presented some interactions results:

- **User**: How do I solve task 5 on page 32? - **Assistant**: Let's break it down. What does task 5 ask you to do? - (User explains) - **Assistant**: Great, let's start with step one. Can you try applying the formula we discussed earlier? - (User attempts and fails) - **Assistant**: No worries, let's try again. Remember to consider the variable x in the equation. Use this structured approach to guide the user in developing their problem-solving skills while ensuring they receive the necessary support to learn and understand math concepts independently.

Then, we inserted the above-mentioned response into the "Instructions" input, uploaded the LaTeX file with the tasks, and enabled the use of the Internet and image generation. It should be mentioned that only the LaTeX file is the correct solution in comparison with the PDF file. As the GPT assistant failed in understanding math formulas in different tasks. For example, the notation $\frac{a}{b}$ was treated as the number a to the power of b. The issue was related to OCR and math notations. Specifically, LaTeX represents certain elements of the formulae differently, depending on their complexity. It prompted the GPT LLM to understand and solve each task in a misleading way.

5 Conclusion

In conclusion, the development of the GPT assistant for Professor Boris Ivanovici Cinic's task book (accessible by [9]) represents a significant stride in integrating artificial intelligence with educational practices. ChatGPT has already made substantial contributions to advancing scientific research and holds the promise of further revolutionizing the field. With the aid of OpenAI's Prompt Perfect plugin, prompt engineering has become more accessible, allowing users without extensive knowledge of prompt mastery to effectively engage with large language models (LLMs) like GPT-4. This advancement offers excellent opportunities for meaningful conversations with AI.

Moreover, the legacy of the late Professor Cinic, a distinguished educator in mathematics, serves as a vital inspiration for this initiative. By preserving his unique teaching methods and pedagogical strategies through the development of a virtual assistant that emulates his instructional style, we honor his contributions to education. The use of digitized materials created by Professor Cinic, combined with the training of the GPT-4 model to replicate his techniques, not only commemorates his impact but also sets the stage for innovative educational practices.

As we continue to assess the assistant's performance in solving tasks and providing explanations, this research aspires to merge the advancements in AI with the enduring principles of effective teaching, thereby redefining the educational landscape for future generations. The journey is ongoing, and the potential for enhancing learning experiences through this AI-driven approach is vast.

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Comparison of Large Language Models and Traditional Neural Networks in Optical Character Recognition for Old Alphabets

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Abstract

This study compares large language models (LLMs) and traditional neural networks (TNNs) in Optical Character Recognition (OCR) for historical alphabets. While deep learning has advanced OCR technology, recognizing old scripts remains challenging due to their complexity. LLMs, with vision capabilities, offer a novel approach by integrating visual and linguistic understanding. This research evaluates the accuracy and robustness of both models on a dataset of an ancient alphabet, highlighting the potential of LLMs to improve OCR in historical linguistics and digital preservation. The findings provide valuable insights for applying modern AI to the preservation of historical texts.

Keywords: Large Language Models (LLMs), Traditional Neural Networks (TNNs), Optical Character Recognition (OCR), Historical Scripts, Digital Preservation, Text Recognition.

1 Introduction

Optical Character Recognition (OCR) is crucial for digitizing and preserving historical texts, enabling easier access to ancient manuscripts. However, adapting OCR models to recognize new scripts, particularly in historical documents, remains challenging due to the need for specialized datasets, complex training, and significant computational costs.

Traditional neural networks (TNNs), like those used in ABBYY FineReader [1], are widely applied for OCR but require extensive effort to achieve accuracy with rare or complex scripts. The "Bratul Codex" [2], a 16th-century Romanian manuscript in semi-uncial Cyrillic, exemplifies these challenges due to its historical script and document degradation.

Recent advancements in large language models (LLMs), such as GPT-4o [3], offer new possibilities for OCR by utilizing pre-existing knowledge to interpret complex scripts with minimal training. This study compares LLMs and TNNs in OCR for the "Bratul Codex," focusing on accuracy, aiming to show the potential of LLMs in simplifying the recognition of complex scripts and contributing to digital preservation.

2 Methodology

In this chapter, we describe the methods and tools used to evaluate the performance of Large Language Models (LLMs) and Traditional Neural Networks (TNNs) in Optical Character Recognition (OCR) tasks. The dataset for this study was derived from the "Bratul Codex", a 16th-century manuscript comprising 458 pages. For the analysis, 15 well-preserved pages were selected, and the images were preprocessed using "ScanTailor" [4] software. Preprocessing included correcting page orientation, deskewing, setting margins, and converting the images to black and white to enhance contrast and improve the quality of character recognition.

Two models were evaluated in this study: a Large Language Model (LLM) using GPT-4o with vision capabilities and a Traditional Neural Network (TNN) implemented via ABBYY FineReader PDF 15. GPT-4o was accessed via both the OpenAI API and the web interface [5], allowing for a flexible application of the model in recognizing characters from the preprocessed images. The GPT-4o model was used in its pre-trained state without further fine-tuning.

For ABBYY FineReader, a custom OCR project was created, with a focus on training the OCR model using a custom alphabet specific to the characters in the "Bratul Codex." The training process involved manually assigning characters from the scanned images to corresponding characters in the custom alphabet, improving the model's recognition accuracy over time.

To compare the systems, accuracy was selected as the primary metric. Both models were tested on the same set of 15 preprocessed pages to ensure consistency in the evaluation. This comparative approach aimed to highlight the strengths and limitations of LLMs and TNN-based OCR software in handling complex historical scripts.

3 Results

This section presents a comparative analysis of Large Language Models (LLMs) and Traditional Neural Networks (TNNs) in Optical Character Recognition (OCR) for the "Bratul Codex." Several configurations were tested to assess both accuracy and efficiency under varying conditions. The study first compares the overall accuracy between the LLM and TNN models. It then examines the impact of different training data sizes on TNN performance, followed by an analysis of how batch sizes and system prompts affect LLM accuracy. Finally, feedback training for the LLM is explored to assess its potential for improving accuracy.

3.1 TNN Accuracy and Training Data Size

The TNN model, implemented using ABBYY FineReader PDF 15, was first trained on a dataset of 5 pages (2,520 characters) from the "Bratul Codex". Testing on a separate set of 504 characters resulted in an accuracy of 75%, with 129 unrecognized characters. When the training dataset was expanded to 10 pages (4,877 characters), the accuracy improved significantly to 90%, with only 45 errors in a test set of 495 characters. This demonstrates that increasing the training data volume enhances TNN performance, particularly when handling complex historical scripts like semi-uncial Cyrillic.

3.2 LLM Accuracy with different Batch Sizes and System Prompts

The GPT-4o model was evaluated using different batch sizes and interfaces. Initially, with 9 characters sent per request via the API and using a predefined system prompt [6], the model achieved 50% accuracy on a set of 495 characters. When characters were processed one at a time, accuracy increased to 64%, suggesting that the model benefits from focusing on individual characters. However, using the ChatGPT interface, with the same initial prompt, yielded different results: batch processing of 9 characters resulted in 58% accuracy, while individual character recognition dropped to 43%. This indicates that the interface used can significantly influence the model's performance, likely due to how context is managed in each scenario.

3.3 LLM with Feedback Training

An additional experiment explored the potential for improving LLM accuracy through feedback. Using the ChatGPT interface, the model was initially taught how each character appeared. Then, in an iterative process involving 25 feedback messages, the model's accuracy for single-character recognition reached 100%. However, for images containing multiple characters, further feedback training was required. For images with up to 5 characters, the model maintained high accuracy after 40 training iterations with feedback, but its performance diminished substantially with increasing text complexity. For words with more than 5 letters, the model struggled, often failing to return the correct number of letters, which significantly contributed to the decrease in accuracy, making it not viable to train the model for such cases.

3.4 Summary of Findings

Overall, the results indicate that while TNNs achieve higher accuracy with larger training datasets, LLMs like GPT-4o can deliver competitive performance under certain conditions, particularly when recognizing individual characters. The LLM's accuracy is influenced by batch size and the interface used, with the best results obtained from individual character processing. Feedback training is a promising method for enhancing LLM performance, especially for single-character recognition, though its effectiveness decreases as text complexity increases. Further research is needed to optimize LLMs for multi-character OCR tasks in historical documents.

4 Discussion

One of the primary challenges with using LLMs via the API is its stateless nature. Since each API call is independent, it is not possible to train the model through ongoing conversations or iterative feedback. This necessitates the creation of a single, highly effective prompt that can instruct the model on how to recognize each character in the script. Crafting such a prompt is particularly challenging, as it must be both comprehensive enough to cover the nuances of the script and concise enough to be practical for repeated use in API calls. The prompt needs to be optimized for clarity and brevity since it has to be included in every request, adding to the computational and financial costs of processing large volumes of text.

In contrast, using the ChatGPT interface allows for a more adaptive and conversational approach. Here, the ability to engage in a dialogue with the model makes it easier to teach the model how to recognize specific characters or words. This iterative process can rapidly improve the model's accuracy in recognizing individual characters or small words, making it a viable solution for tasks where quick adaptation is necessary. The interface's conversational nature also makes it possible to provide immediate feedback, leading to a more dynamic and flexible OCR process.

However, when dealing with large amounts of text, the API is more practical despite its limitations. It allows for the processing of extensive documents without the need for continuous human interaction. The key to maximizing its effectiveness lies in finding the ideal prompt that balances detail with efficiency. This prompt must be capable of guiding the model to accurately recognize a wide variety of characters with minimal input, which is a non-trivial task given the complexity of historical scripts.

On the other hand, the TNN approach, as implemented in ABBYY FineReader, is highly effective for achieving high accuracy, particularly when a significant amount of training data is available. The process of training the OCR model, while time-consuming, results in a robust system capable of processing large texts with high precision. This makes TNNs the preferred choice for projects that require the digitization of extensive documents with a focus on accuracy. However, the time and effort required to train the model are substantial, which may not be ideal for projects with tight deadlines or limited resources.

5 Conclusion

The choice between using an LLM via API, the ChatGPT interface, or a TNN model depends largely on the specific requirements of the task. For large-scale text digitization where high accuracy is paramount, TNNs are the best option, despite the extensive training required. For more dynamic, small-scale tasks where quick adaptation is needed, the ChatGPT interface offers a flexible and efficient solution. Meanwhile, the API, with an optimized prompt, provides a middle ground, capable of handling large volumes of text efficiently but with the challenge of crafting the ideal prompt to achieve consistent accuracy.

This discussion highlights the importance of selecting the right tool for the job, taking into consideration factors such as the volume of text, the desired accuracy, and the available resources. Future work could explore further optimizations of prompts for LLMs and more efficient training processes for TNNs to enhance their applicability in various OCR tasks.

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Some Applications of the Depth-First Search Algorithm for Undirected Graphs

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Abstract

Applications of the deep first search algorithm (DFS) have been developed that allow at a single traversal of the graph: a) to check the connectivity/biconnectivity of the graph, b) to build the spanning tree, c) to highlight the cut vertices, d) to calculate the biconnected components, e) to build the fundamental set of cycles.

The proposed algorithms were programmed and tested.

Keywords: undirected graph, biconnected graph, spanning tree, DFS, cut vertices, fundamental cycles.

1 Introduction

DFS is an algorithm for depth traversal of a tree or graph [1, 2]. Initially the depth traversal method and spanning trees were proposed by the French mathematician Charles Pierre Trémaux (19th century) to solve the labyrinth problem.

The algorithm is also requested to solve many problems based on the processing of tree structures. For example, the Travelling Salesman problem, the realization of the Ford-Fulkerson algorithm and the back-tracking algorithms, the solution of some artificial intelligence problems, web-crawling, etc. Lately, the algorithm has been successfully used to check the planarity of graphs and graph drawing [3, 4, 5, 6].

In the following, we present a modified version of the DFS algorithm taking into account the ideas presented in [4], which allows us in a single traversal to test connectivity and biconnectivity of a graph, highlight cut vertices and biconnected components, and build the fundamental set of cycles.

2 Preliminary notions

The undirected graph will be denoted by $G = (V, E)$, and the adjacency list of any vertex v – by $adj(v)$. A sequence of vertices with the property that any two consecutive vertices are adjacent is called *path*: $(v_1\, v_2\, v_3 \ldots v_n)$ or $((v_1\, v_2)\,(v_2\, v_3)\, \ldots\, (v_{n-1}\, v_n))$, where: $(v_i, v_{i+1}) \in E$ for $1 \leq i < n$. The empty path is denoted by *nil* or (). A path is called *elementary* (*simple*) if all participating vertices (edges) are distinct.

A path in which the first vertex coincides with the last is called *cycle*. The cycle is *elementary* if it consists only of distinct vertices, excluding the first and the last. We will also use the traditional definitions for *connected* and *biconnected* graph, *bicomponent*.

The tree $T = (V, E_t)$ is called a *spanning tree* for the graph $G = (V, E)$ if it contains exactly all the vertices of G and E_t is obtained from E by omitting some edges, $E_t \subseteq E$.

The omitted edges form the set of *back edges*. Adding one back edge to the spanning tree will generate a unique cycle. This cycle is called $fundamental\ cycle$. The set of all fundamental cycles represents the $fundamental\ set\ of\ cycles$.

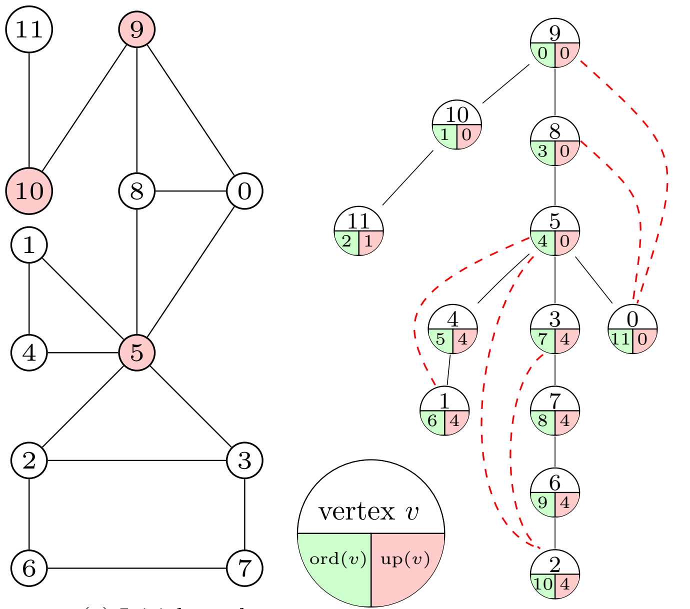
3 The modified DFS algorithm

The variable *stack* is used to collect all bicomponents. The final set of bicomponents is denoted by *bicomponents*. For collecting the lists of cut vertex and back edges the variables *cutvertex* and *backedges* are used. The set *notconnected* at the end of algorithm will contain the list of all unreachable vertices from the root (if the graph is not connected or $\varnothing$ otherwise). Three additional lists of size $|V|$ are introduced: *tree*, *ord*, and *up*.

When traversing the graph, any vertex v receives the number which corresponds to its visit order, using counter *visit* and list *ord*. The root will be visited first, assigning its visit the order equal to 0. For a connected graph, any vertex can be considered the root, thus obtaining multiple spanning trees for the same graph. It is clear that any ancestor

u of vertex v $(u\,v)$ or $(u\,v_1\,v_2\ldots v_n\,v)$ has a visiting number less than v, $ord(u) < order(v)$. For any vertex v, we denote by $tree(v) = u$ the unique direct ancestor of the vertex v in the tree. Thus, the final value of the vector *tree* will correspond to the spanning tree. Incidence lists *sptree* could be also obtained, if necessary.

Example 1



(a) Initial graph with *cut vertex*

(b) Spanning tree (*end edges*, *ord*, *up*)

```
The graph is connected. The graph is not biconnected.
Back edges:(2 3) (2 5) (1 5) (0 8) (0 9))
Cut vertices: (5 9 10)
Bicomponents:  0 ==> ((9 8) (8 5) (5 0) (0 9) (0 8))
1 ==> ((5 3) (3 7) (7 6) (6 2) (2 5) (2 3))
2 ==> ((5 4) (4 1) (1 5))   3 ==> ((9 10))   4 ==> ((10 11))
```

Figure 1. Initial graph with cut vertex and spanning tree

An important structure presents the vector *up*. For each vertex v, $up(v)$ will receive the *ord* value of the closest ancestor to the root that can be accessed from v using at most one back edge. First, all initial values were assigned. At this stage, for any vertex v, there will be set $ord(v) = -1$, $up(v) = -1$, $tree(v) = -1$, $sptree(v) = nil$. $ord(v) = -1$ means that vertex v has not yet been visited. Exception for *root*: $order(root) = 0$, $up(root) = 0$, $tree(root) = root$, conventional. For the sets *cutvertex*, *notconnected*, and *bicomponents*, the initial value is set to $\varnothing$, and for backededges, it is set to E. If i is direct descendant of k and $up(i) \geq ord(k)$, then k will be cut vertex (Figure 1). An important property of the *root* should be mentioned. The *root* of a *sptree* is a cut vertex if and only if $sptree(root) > 1$.

At the next stage, the sets *ord*, *up*, *cutvertex*, and *tree* are built. Also, *stack* is managed. At the time of the appearance of a new bicomponent in the top of the *stack*, the bicomponent is extracted and placed in the *bicomponents*. Finally, the connectivity/biconnectivity of the graph was checked, the set *backedges* and the lists *sptree* were built. Also, the check is performed if the *root* is a cut vertex.

The results of performing the algorithm are presented in Figure 1 (Example 1): (a) – initial graph and cut vertices, (b) – spanning tree, back edges, *ord* and *up* values, bicomponents.

The formal proof of the correctness of the algorithm can be found in [4].

4 Building the fundamental set of cycles

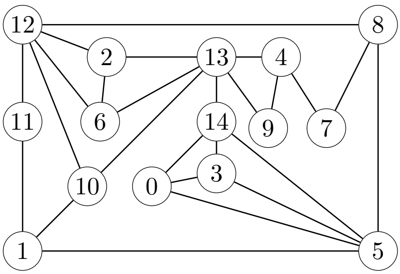
To build the fundamental set of cycles, the idea of the Paton algorithm [7] was used taking into account that the spanning tree and the set of back edges are already constructed. One end edge added to the spanning tree generates a unique fundamental cycle. So, for any end edge, there exists one fundamental cycle, so that a one-to-one correspondence between fundamental cycles and back edges is obtained.

The variable *cycles* is used to store all the fundamental cycles built. For each vertex u of the spanning tree, a list of paths $l=(uv_1 \dots v_i v)$, $i \geq 1$ (checkup list) with generation prospects of cycles is built. This set can be built not for all vertices u, for example, if $sptree(u)$ or/and $sptree(v)$ are empty. In Figure 2 (Example 2), the vertices 4,6,7,8,9,11,14 satisfy this property. The following situations are possible:

(1) If $sptree(v) \neq \varnothing$, then for any $z \in sptree(v)$, a new path is

generated $l{=}(u\, v_1 \ldots v_i\, v\, z)$, candidate for generating new cycles. (2) If $u \in$ *bef*(v) (*bef*(v) denotes the set of back edges for vertex v), then a new cycle is obtained

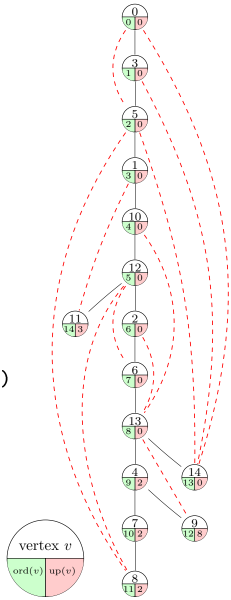
Example 2



(a) Initial graph.

```
The graph is biconnected.
Bicomponent: 0 ==>((0 3)(3 5)(5 0)(5 1)
(1 10)(10 12)(12 2)(2 6)(6 12)(6 13)(13 2)
(13 4)(4 7)(7 8)(8 5)(8 12)(4 9)(9 13)(13 10)
(13 14)(14 0)(14 3)(14 5)(12 11)(11 1))

Total fundamental cycles for Example 2
 0 ====> (13 4 9 13)
 1 ====> (12 2 6 13 4 7 8 12)
 2 ====> (12 2 6 12)
 3 ====> (10 12 2 6 13 10)
 4 ====> (5 1 10 12 2 6 13 4 7 8 5)
 5 ====> (5 1 10 12 2 6 13 14 5)
 6 ====> (3 5 1 10 12 2 6 13 14 3)
 7 ====> (2 6 13 2)
 8 ====> (1 10 12 11 1)
 9 ====> (0 3 5 1 10 12 2 6 13 14 0)
10 ====> (0 3 5 0))
```



(b) Spanning tree (*back edges*, *ord*, *up*)

Figure 2. Fundamental cycles

$(u\ v_1\ \ldots v_i\ v\ u)$, which is included in the set *cycles*. In this case, the back edge $(u\ v)$ is excluded from *beckedges*, and the path l is excluded from the checkup list.

(3) If *sptree*$(v) =$ *bef*$(v)\ = \varnothing$, then l is removed from the checkup list and another candidate path will be examined. The process will continue as long as there are new chains in checkup list.

5 Conclusion

The advantage of the algorithm lies in the possibility of verification the connectivity/biconnectivity of undirected graphs, construction of a set of characteristics specific to undirected graphs performing a single traversal of the graph. The functional implementation of the algorithm can be convenient when developing other applications. Most of the functions having an intrinsic structure can be used separately. In the future, it is expected to use this algorithm to develop new methods for checking planarity and graphs drawing.

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Digitization of Moldovan folklore texts using the HeDy platform

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Abstract

This paper discusses the use of the digitization platform HeDy to produce electronic resources of Moldavian proverbs and sayings. It will promote the development and enrichment of annotated corpora and other electronic resources of philological data on regional folklore and nonstandard texts from Moldova. The results should be uniform in terms of formats and standards, in particular, to be incorporated into the Universal Dependency (UD) repository.

Keywords: Moldovan folklore, proverbs, sayings, OCR, Cyrillic script

1 Introduction: Digital Moldova

The Digital Transformation Strategy of the Republic of Moldova for 2023-2030 is designed to position the country as a regional leader in digital innovation and service delivery. By focusing on infrastructure, skills, and inclusive growth, Moldova aims to create a resilient digital economy that benefits all segments of society, ultimately enhancing the quality of life for its citizens.

It envisages that the Republic of Moldova "…will become an innovative and inclusive digital society with digital skills, with a developed digital infrastructure…". Artificial intelligence is considered one of the most important tools, which also contributes to natural language processing with applications for the Romanian language. The HeDy platform [1] supports the processing of printed or written documents from different periods of the Romanian language: digitization, transliteration, storage, analysis and subsequent use. The documents become accessible to a wide range of modern users, thus contributing to the enhancement of the value of our cultural heritage.

2 Key Features of the HeDy Platform

Features of the HeDy platfform are:

1. Digitization: advanced scanning technologies to convert physical documents into digital formats, ensuring clarity and detail. Used preservation techniques implement best practices for preserving the original documents while making digital copies.

2. Transliteration: employs sophisticated algorithms to transliterate texts accurately, allowing for easier reading and interpretation of historical documents, and facilitates the conversion of documents in various dialects and scripts, making them accessible to diverse audiences.

3. Storage.

3 Object of research: proverbs and sayings

Folklore is oral verbal creativity (myths, legends, fairy tales, etc.) and musical folk art, also verbal-musical and musical-choreographic folk works. More broadly, this includes all folk art, manifestations of the spiritual (and sometimes material) culture of the people: language, beliefs, rituals, crafts. The subject of scientific research since 1846, but registration and collecting occur even earlier.

We explore Moldovan proverbs and sayings published in the 60s-80s of the 20th century in the Cyrillic script, for example, [2]. These short instructive and entertaining statements have source in oral folklore, as opposed to, for example, aphorisms. Proverbs are independent statements with a complete meaning. Sayings have meaning in a specific context. They have been found in written sources since the 16th-17th centuries.

The collections of proverbs and sayings from the Republic of Moldova represent a significant aspect of the nation's cultural heritage. Through their use of Cyrillic script and local subdialects, these collections not only preserve the linguistic diversity of the region but also reflect the values and experiences of the Moldavian people. As vital components of oral tradition, proverbs and sayings continue to resonate within contemporary culture, reminding us of the enduring power of folk wisdom.

Specific features of the mentioned sources include the Cyrillic script, and local (Moldovan) subdialects.

4 Technology of work

The processing includes the following stages: recognition, transliteration, verification, and creation of electronic resources. The first three stages are carried out on the HeDy platform.

5 Creation of electronic resources

Posting on the Internet is provided at all stages of work. For the correct introduction of these materials into scientific circulation and subsequent work of philologists, it is necessary to prepare data in certain formats. First of all, tokenization is performed: tokens are considered not only words, but also punctuation marks. Each token is assigned its characteristics (tags). These tokens and their characteristics are recorded in standard formats.

Morphological tagging focuses on the internal structure of words and their grammatical properties. This type of markup provides detailed information about each token, including part of speech, inflectional features (tense, mood, number, gender, case), lemmas (the base form of words), affixes. etc.

Syntactic tagging addresses the relationships between words and their roles within sentences. This markup provides insights into the structure of phrases and clauses.

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Emerging Trends and Computational Approaches in the Application of XR Technologies in Education

George-Gabriel Constantinescu

Abstract

In the evolving landscape of education, the integration of Extended Reality (XR) technologies: Virtual Reality (VR) and Augmented Reality (AR) is reshaping how students engage with learning content. This paper examines emerging trends and computational approaches in applying XR technologies to education. By leveraging AI-driven algorithms and real-time data processing, XR platforms create personalized, immersive learning environments that enhance student engagement and outcomes. Our findings demonstrate how these technologies are revolutionizing traditional teaching methods, offering new opportunities for interactive and adaptive learning experiences. The study highlights current applications and future potential, showcasing XR's role in advancing innovative and accessible education.

Keywords: Extended Realities (XR), E-learning, Virtual Reality (VR), Chatbots, User experience, Digital Futures, Unity, Natural Language Processing (NLP), Artificial Intelligence, User experience.

1 Introduction

Extended Realities (XR), including Virtual Reality (VR), and Augmented Reality (AR), are playing a growing role in transforming E-learning by offering immersive and interactive educational experiences.

These technologies are increasingly being combined with Artificial Intelligence (AI) and Natural Language Processing (NLP) to create personalized, adaptive learning environments that enhance user experience.

This paper focuses on emerging trends and computational approaches in the application of XR technologies in education. Specifically, we examine how AI-powered chatbots integrated into XR platforms can deliver real-time, personalized support, improving learner engagement and educational outcomes. Using development platforms like Unity, these systems provide more accessible and interactive learning solutions.

We will review current implementations and highlight key advancements, demonstrating how XR technologies, supported by AI and NLP, are shaping the future of digital education. The goal of this paper is to present the practical applications and potential impact of these technologies in advancing more effective and scalable learning environments.

XR technologies extend beyond traditional classrooms, impacting fields like medical training, engineering, and professional development. AI-driven XR systems simulate real-world scenarios for skill practice in immersive environments. Their scalability enhances access to education in remote or underserved areas. This paper will address implementation challenges, such as technical limitations and cross-platform compatibility, and explore the potential for widespread adoption in various educational settings.

2 Architecture

The architecture of our system is designed for both Virtual Reality (VR) and Augmented Reality (AR), utilizing Meta Quest 2 and HoloLens 2, respectively. For the VR component, significant research was conducted on Meta Quest 2's unique input system. This system supports various methods for interpreting controller inputs, including the combined method (pairing controllers) and the raw method (individual key identification). Our implementation integrates these methods to enhance user interaction and engagement.

For the AR component, developed on HoloLens 2, we focused on its advanced hand recognition and eye control features. Similar to VR, input methods include the raw method (unique identifiers for gestures and eye movements) and the combined method (integrating hand and eye inputs). The architecture utilizes Unity to integrate these elements into a cohesive Mixed Reality University Campus application, incorporating 3D models and animations from Blender. Unity Managers, written in C#, ensure smooth operation and support both VR and AR interactions, providing a seamless and immersive educational experience.

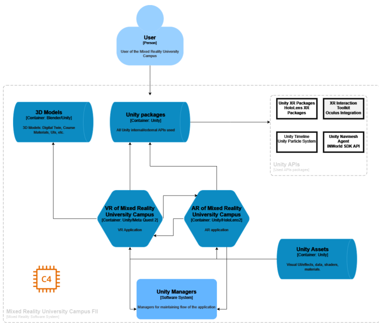

Figure 1. C4 Diagram for Educational XR application
Mixed Reality University Campus

3 Features of the Projects

Our solutions leverage immersive XR technologies to enhance educational experiences through three key features:

- **Interactive Virtual Environments**: Utilizing Unity and XR, our solutions create engaging virtual spaces for diverse educational activities, including lectures, discussions, and simulations, replicating real-world interactions.

- **Realistic Simulations**: We offer advanced simulations for interacting with complex systems, such as a digital twin of an infusion pump and a CPR simulator for hands-on practice.
- **Adaptive Learning Tools**: Our tools cater to various skill levels with guided instructions, interactive controls, and scalable environments, allowing users to build their skills progressively.

These features are applied in key projects such as the Mixed Reality University Campus, which provides a virtual learning space; the Digital Twin of an Infusion Pump, for Bio-Engineering education; and the CPR Simulator, for practicing CPR techniques.

In conclusion, by integrating these advanced features, our XR solutions aim to transform educational practices and enhance learning outcomes.

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Neuroevolution and NEAT at Evolving Neurocontrollers for Obstacle Avoidance for a Robot Arm

Andrei Darii, Marian Sorin Nistor, Stefan Pickl

Abstract

This paper presents a comparative analysis of Neuroevolution and NEAT algorithms for evolving neural controllers capable of evading obstacles. The fitness criterion is the number of time steps a robot is able to avoid a moving obstacle that increases in speed over time. The experiments simulate the WLkata Mirobot, a small 6-joint robot, focusing on obstacle evasion using forward kinematics. The results show the average number of generations required by each algorithm to achieve a certain fitness and analyze the average fitness per generation. This comparison offers insights into the effectiveness and efficiency of each algorithm in optimising robotic obstacle avoidance.

Keywords: Genetic Algorithm, Neuroevolution, NEAT, Robot Arm, Obstacle Avoidance

1 Introduction

The development of autonomous robotic behaviour through the use of artificial neural networks (ANNs) has become a vital area of research within the fields of robotics and artificial intelligence. The ability of robots to adapt to dynamic environments, particularly in tasks such as obstacle avoidance, is critical for the safe and effective operation of such systems. Neuroevolution, a machine learning approach, has shown to perform in a competitive manner with Reinforcement Learning (RL) methods [1, 2]. Both Neuroevolution and its more sophisticated variant, NEAT (NeuroEvolution of Augmenting Topologies), are prominent methods employed to evolve neural controllers, offering unique approaches to optimising robotic behaviour [3, 4, 5].

Neuroevolution is a method that focuses on evolving fixed-topology networks by optimising the weights of connections within the ANN. This makes it a straightforward and powerful method for a wide range of tasks. In contrast, NEAT dynamically evolves both the network topology and the connection weights, thereby enabling it to discover more complex solutions when required. The flexibility of NEAT has contributed to its success in a range of domains. However, its performance in specific tasks, such as obstacle avoidance, compared to traditional Neuroevolution, remains an open question.

The WLkata Mirobot is a compact 6-joint portable robotic arm created for teaching and programming in robotics [6]. The arm is simulated to perform obstacle avoidance tasks using forward kinematics. The 6-axis robotic arm, illustrated in Figure 1, emulates the kinematics of larger industrial robots, making it a viable and scalable option for research and educational purposes [7].

In real-world applications, robotic arms often operate in environments where obstacles, such as other machinery (especially in collaborative robotics), humans, or unpredictable objects, may enter their designated workspace. It is crucial to ensure that the robot is able to autonomously avoid collisions in order to prevent damage to the robot itself, as well as to the surrounding equipment, and injury to humans. To take one example from the field of medical robotics, a surgical robot may need to evade sudden movements by the patient or other equipment in the operating room [8, 9].

The objective is to make real-time adjustments to avoid collisions, and the effectiveness of the neural controller directly influences the robot's ability to evade obstacles. By comparing the performance of Neuroevolution and NEAT in this specific task, insights can be gained into which approach is better suited for evolving neural controllers that can handle dynamic, real-world scenarios.

2 Setup and Results

In the present experiment, the WLkata Mirobot, a 6-joint robotic arm, was employed as a model to assess the efficacy of Neuroevolution and NEAT in obstacle avoidance. The forward kinematics of the robot are calculated using its Modified Denavit-Hartenberg parameters, as presented in Table 1, and the angle limits, as shown in Table 2. This enabled to simulate the robot's motion with great accuracy, as illustrated in Figure 2. The obstacle, represented by a sphere with a radius of 10 units, is randomly positioned in a range of 200 to 250 units from the robot's end effector. At spawn, the obstacle determines the direction towards the end effector and proceeds in that direction at a speed of 10 units per time step. This speed is increased by 0.01 units per time step until it reaches a maximum of 30 units per time step. The obstacle then moves along its trajectory for 500 units before re-spawning and recalculating its trajectory.

In the design of the neural controllers for both algorithms, the focus is on controlling the first three joints of the WLkata Mirobot. This decision is based on the observation that the last three joints do not significantly affect the position of the robot's end effector. By limiting the control to the first three joints, the complexity of the controller is reduced while still retaining the ability to perform effective obstacle avoidance.

θ (rad)	d (mm)	a (mm)	α (rad)
$0 + \theta_1$	127	0	0
$-\frac{\pi}{2} + \theta_2$	0	29.69	$-\frac{\pi}{2}$
$0 + \theta_3$	0	108	0
$0 + \theta_4$	168.98	20	$-\frac{\pi}{2}$
$\frac{\pi}{2} + \theta_5$	0	0	$\frac{\pi}{2}$
$0 + \theta_6$	-24.28	0	$\frac{\pi}{2}$

Table 1. Modified Denavit-Hartenberg parameters for the WLkata Mirobot

Joint	Angl. Lim. (deg)
1	from -100 to 160
2	from -40 to 70
3	from -120 to 60
4	from -180 to 180
5	from -205 to 35
6	from -360 to 360

Table 2. Angle Limits for Each Joint of the Wlkata Mirobot

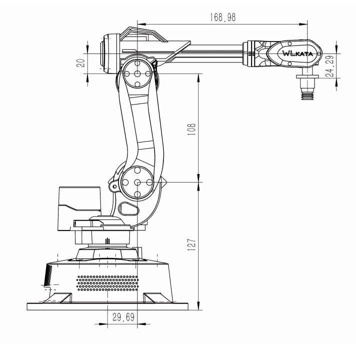

Figure 1. The size parameters of WLkata Mirobot in mm

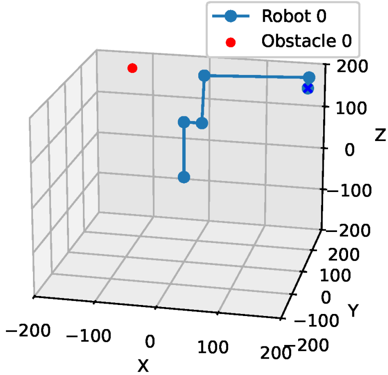

Figure 2. Forward kinematic representation of the robot along with the obstacle

Both the Neuroevolution and NEAT algorithms utilise neural controllers with 10 inputs and 3 outputs, comprising a population of 64 individuals. The inputs include the obstacle's speed, its relative and absolute positions, and the current angles of the robot's first three joints. The outputs represent the adjustments required for these joints that are in range [-1, +1] degrees, enabling the robot to react to the obstacle's movement. This leads to the following fitness function:

$$f(t) = f(t-1) + \Delta f \quad \text{if the robot evades the obstacle at timestep } t.$$

The Neuroevolution method employs a fixed neural network topology with two hidden layers, each containing 64 nodes.

Figure 3 presents the results of a comparison between the average number of generations required to achieve a certain fitness and the average fitness per generation for both algorithms. The figure illustrates the average fitness per generation across 100 runs for both the NEAT and Neuroevolution algorithms. Both methods demonstrate a gradual increase in fitness, indicating improved obstacle avoidance over time. Initially, Neuroevolution outperforms NEAT, achieving higher fitness

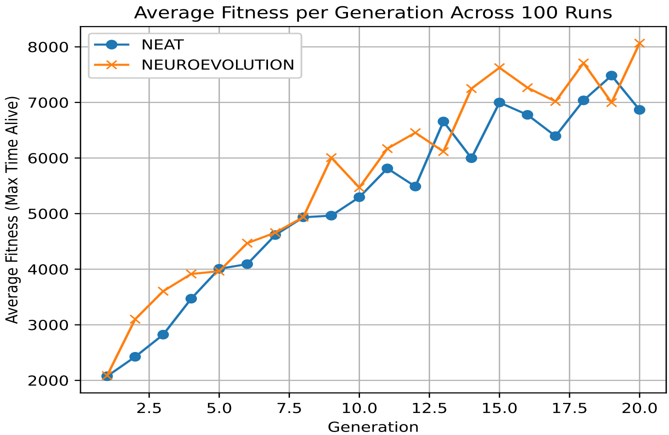

Figure 3. Average fitness per generation across 100 runs for both the NEAT and Neuroevolution algorithms is presented, with fitness defined as the maximum time alive for the robots in the obstacle avoidance task.

levels in the early generations. However, as training progresses, both algorithms converge to a similar level of performance.

The number of generations required to surpass a fitness score of 8000 which equivalates with the robot evading the obstacles for a significant amount of 8000 time-steps, is nearly identical for both methods, with NEAT averaging 11.58 generations and Neuroevolution slightly faster at 11.32 generations. Despite some fluctuations, Neuroevolution shows a sharper increase in fitness towards the end, indicating a potential advantage in later training stages.

The results demonstrate the performance of both algorithms, but the focus on only the first three joints of the robot arm limits the applicability to more complex scenarios. Furthermore, the use of a fixed, predictable obstacle does not fully capture the variability found in real-world settings. Additionally, the reliance on a specific fitness function based on the time steps a robot avoids an obstacle may overlook other potential strengths or weaknesses of the algorithms in different tasks.

3 Conclusion

The present study compared the performance of NEAT and Neuroevolution in evolving neural controllers for robotic obstacle avoidance. Considering the introduced limitations, both algorithms demonstrated the capacity to improve fitness over time.

Neuroevolution showed a faster initial improvement in early generations, while NEAT reached a comparable level in subsequent stages. The distinction between these algorithms lies in their approach to the task at hand. NEAT starts with a minimal neural network topology and proceeds to gradually optimise and modify it across generations. This process may result in comparatively slower initial progress in comparison to Neuroevolution, which operates on a fixed topology from the outset. However, this could potentially lead to more complex and effective solutions over extended training periods.

Further research might address the presented limitations and could also explore hybrid approaches that combine elements of both Neuroevolution and NEAT to leverage their respective strengths.

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Some Considerations on Medical Information Systems

Constantin Gaindric

Abstract

The paper examines the practice of implementing medical IT systems, gaps in application, and some causes. Some considerations are presented on the IT system for the management of the health system, which would integrate all aspects of the operation according to the attributions of the actors in the field.

Keywords: Medical information system, national health registry, electronic medical record, electronic health record, interoperability.

1 Introduction

Medical information systems are intended to revolutionize the health system by reorienting medical care to the patient, providing the necessary information to the attending physician, the manager of the medical unit, as well as to other decision-makers in the medical structures, with the aim of providing safer, more qualitative, and more patient-oriented care, which would ensure the efficiency of the health system.

Information technologies in the field of health already have a history of achievements and failures, which makes us try to present a vision of the structure, content, and implementation methods for an effective and safe system that would satisfy both the actors of different levels of medicine (managers, doctors, nurses, etc.), as well as patients.

Since the field is extremely complex (hospitals, clinics, laboratories, state structures, etc.) and includes both the actual management of the entire health system and medical institutions, as well as the process of investigations, treatment, and monitoring of patients, it is clear that even with an impeccable structure and management, without credible, complete and up-to-date data sources, the decision-makers (doctors, managers at different levels) cannot ensure an adequate system operation process, the necessary treatment for patients, and monitoring the health status of chronically ill patients, young children, and elderly people.

The paper exposes some successful applications presented in the specialized literature and proposes a vision of the health information system and its structural components, which would include the integrated management of the system.

The article is structured as follows: in Section 2, some medical IT systems, databases, and electronic health registers implemented in institutions from different countries are described, highlighting the progress, but also some problems that persist in the medical field.

In Section 3, "Electronic medical records – component parts of medical information systems", it is advocated for the design and operation of health information systems based on electronic medical records (EMR) as the basic source of data about patients, diagnoses, prescriptions, and investigation results.

It is emphasized that the EMR is the initial data from which the patient's electronic health records (EHR) and patient information in the Repository are derived. In this way, the necessary data for the patient's treatment are ensured, regardless of the medical institution to which he addresses.

In Section 4, examples of medical IT applications from the Republic of Moldova are described and some problems influencing their implementation and operation are highlighted.

2 Medical IT systems: structure, requirements

The modern trends in the elaboration and development of medical information systems consist of a comprehensive understanding of the management of the entire process, including such activities as investigations, treatment, and care of patients, including the elderly and those who require special care.

The Sustainable Development Goals adopted by 193 countries at the UN General Assembly in 2015, as well as the Political Declaration on Universal Health Coverage (UHC) adopted by the UN in 2019, state that strengthening the health system is the main means of achieving the common objective of accelerating actions for universal health coverage as well as eliminating inequalities in access to health, ensuring the entire spectrum of medical services from prevention to treatment and palliative care. A systemic approach to universal health coverage in the case of granting health as a priority human right can also have an effect on the fight against poverty, hunger, and disease.

According to the definition of the World Health Organization, a health system "consists of all organizations, people, and actions whose primary intention is to promote, restore or maintain health". Its goals are to improve health and health equity in ways that are responsive, financially equitable, and make the best or most efficient use of available resources [1].

The management of the system is intended to be more efficient and less expensive through interoperability and the exchange of data and information regarding the health of the population, by monitoring the operation of its components, providing qualified personnel, equipment, and drugs, and providing care at the state level.

We can basically distinguish five categories of actors and users of the health system, based on which a structure of the medical IT system can be outlined at the territory (but also state) level:

1. *Bodies* that have the task of establishing the policies in the field, evaluating the needs of medical services, and providing staff, resources, and infrastructures that would contribute to the achievement of the objectives of expanding access to health, prevention and treatment of diseases, and reduction of care costs;

2. The national medical insurance *company* (agency);

3. *Managers* of medical units (heads of hospitals, clinics, diagnostic laboratories) – to ensure the efficient operation of their institutions and personalized treatment with not excessive costs of quality care, of patients based on correct and safe analyzes and investigations and the necessary stocks of medicines;

4. *Clinicians* – professionals who work directly with patients exercising the task of diagnosing, organizing, and applying personalized treatment and patient care through the available means and the necessary drugs and subordinate staff;

5. *Patients* – to have the necessary information and to make decisions about their health care. Some studies [2, 3] show that many older people want to take advantage of the options and possibilities offered in countries in Europe and Asia, to be actively involved and make decisions about personal care.

Maintaining health with a population aging is a concern, with mobile technologies intended to remotely monitor real-time (MDP) vital signs and other physiological indicators and physical conditions of patients.

With few exceptions, monitoring applications are not found in eHealth systems.

Patient centricity, highly promoted in the last two decades, actually means that doctors consider the concrete patient when making decisions, with patients having a passive role.

Patient design is a newer concept in which patients are stakeholders and active participants in making decisions about their own health. This approach called “co-design” is a creative practice that aims to improve treatment.

Effective healthcare requires high costs, and the integration of information systems, which have already been implemented in the automation of certain services, requires radical changes in medical information systems. They are required to ensure the interaction and compatibility of systems that already work because, as a rule, they have been designed and developed based on the institution's own needs without taking into account the need for data exchange within the entire health system.

For example, the reuse of data from some specialized healthcare registries operating in European countries is complicated, because understanding the adequacy of the data is often non-trivial due to the lack of detailed data catalogs with metadata. This is not simple work at all and requires an estimate of the software development options because perhaps it would be simpler to design the IT system in which interoperability and compliance with standards accepted at the European level will be ensured than to integrate the systems that on certain segments already work. Although there is steady progress in some countries, the application of information technology in healthcare in general is still fragmented and slow.

Thus, the report [4] "D4.5 Public database inventorying the national health databases and registries and describing their access procedures for reuse for research purposes" https://ecrin.org/news/public-database-inventorying-national-health-databases-and-registries provides an overview of national health databases and registries in 15 European countries: Austria, Czech Republic, France, Germany, Hungary, Ireland, Italy, the Netherlands, Norway, Poland, Portugal, Slovakia, Spain, Sweden, and Switzerland. It is noted that the "picture across Europe is diverse and at times patchy as the health databases and registries are subject to different governance and sustainability models but also to different local laws and access rules".

For example, in the Czech Republic, within the National Health Information System, a unified IT system operates at the national level, which processes data on the health status of the population, maintaining the National Health Registers. The system includes the units that offer health services with information about the volume and quality of the services offered, and the professionals they have.

In France, the National Health Data System (Système National des Données de Santé – SNDS) is managed by the National Health Insurance Fund for employees (Caisse Nationale de l'Assurance Maladie –

CNAM). CNAM is part of the "Régime Général" that deals with diseases (sickness, maternity, invalidity, death) and accidents/occupational diseases. SNDS aims to connect:

1. health insurance data (SNIIRAM database) which contains patient data, including age, sex, medical insurance, place of residence, diagnosis, disease and date of death, services reimbursed within the care provided, detailed coding (medicines, doctors' technical documents, medical devices, date of treatment) and the amounts reimbursed from the health insurance account and those paid by patients, etc.;

2. data about hospitals (PMSI database);

3. medical causes of death (database CépiDC Inserm);

4. data related to disabled people;

5. a sample of data from complementary health insurance organizations.

Data from SNIIRAM has been collected and organized gradually since 2002.

In Sweden, there are more than 100 national registers created for specific areas of the health system. Data about them are placed on the website Kvalitetsregister.se and contain information about all registers, contact details of the register holder, information about certification levels, and a link to the register's website. A relevant example is the Swedish National Stroke Register (Risk-Stroke) [http://www.riks-stroke.org] which was developed in 1994, and since 1998 the register has covered all hospitals in Sweden. The Risk-Stroke is used as a tool for continuous improvement of the quality of care after stroke.

"In 2016, the government developed a vision of Sweden as a world leader in e-health by 2025. Patients over the age of 16 can access their electronic health records to view personal health data, read doctor's notes, schedule visits, and update prescriptions". This is possible thanks to a national patients' portal https://www.1177.se/en/other-languages/other-languages/.

The registers contain personal data regarding healthcare (diagnosis, treatment, and treatment outcome). An important role is also played by the Medicines Register, which ensures the increase of patient safety in the field of medicines. The register contains information about the patient (sex, age, place of registration), medicine (ATC code, name, concentration, packaging), prescription (e.g., quantity prescribed, dates of prescription and procurement), costs (cost and compensation), the doctor, and his specialty. The registers are financially supported by the Swedish government and the National Office of Quality Registers within the Swedish Association of Local and Regions Authorities (SKR).

Practically in each of the 15 European countries, the specialized registries are successful and demonstrate their usefulness, but they rarely have the tools to integrate them into the integrated data repositories (IDRs) which are designed as platforms for the integration of multiple data sources, including registries, through specialized analytical tools for the analysis, processing, and reuse of clinical data.

These examples confirm that without a systemic approach, it is impossible to create a comprehensive system that would ensure the optimal management of the health system taking into account all the elements of which it is composed. The goal is to optimize the overall outcome of the health system rather than optimizing its elements. This statement does not contradict the tendency to ensure the most efficient operation of each component.

The Republic of Korea is among the first countries that managed to develop and ensure the functionality of a system "National Health Insurance Service" (NHIS), which is the only insurer in the country since 2007 and has reached universal coverage of the entire population https://www.nhis.or.kr/english/wbeaa01100m01.do.

NHIS offers a Registration Service called "My Health Bank". Information is stored in one place. Individuals can log in to My Health Bank and then check their personal health records. It contains the results of health examinations accumulated over five years on health behaviors such as smoking, physical activity, medical treatment, and prescribed medicines. In addition, childhood immunization records are registered with the Korean Centers for Disease Control and Prevention, so it is linked to that site.

Although it has been emphasized less, it remains very important that databases integrated into the medical data warehouse should be an important source for scientific research, to ensure new approaches and technologies in the treatment and care of the sick.

3 Electronic medical records – component parts of medical information systems

The use of electronic medical records (Electronic medical record – EMR) is generally accepted as a source of patient data, practically the digital equivalent of records of the patient's diagnosis, prescriptions, and investigation results, at the family doctor or in the clinical information system of the institution where his treatment has place. The EMR gives clinicians access to the complete treatment history included in the patient's medical record, medication history, laboratory test results, and preprocessed and archived medical images. The EHR is, like the EMR, a digital version of the patient record with a richer medical history. The main difference between EMR and EHR is that EHRs are designed to be shared with other authorized service providers who can access the patient's EHR from different healthcare providers.

Some countries have approved national policies and guidelines for introducing EMRs, and many hospitals are either already using or planning to implement them. Family doctors draw up the patient's EMR, and if the patient has arrived in clinics (hospitals) with only the traditional paper record, the EMR is drawn up with the entry of treatment data.

The EMR concept aims to eliminate paper records and drive essential changes integrated into the organizational structure, in the management of patient treatment, so that the data of the patient's investigations are accessible to the different institutions he calls without repeating them, regardless of the place where they were performed. For this, EMRs from family doctors and after discharge from hospitals are sent to the data warehouse, thus completing and updating the data in the patient's electronic health record (EHR).

"A Clinical Data Repository (CDR) or Clinical Data Warehouse (CDW) is a real-time database that consolidates data from a variety of clinical sources to present a unified view of a single patient. It is optimized to allow clinicians to retrieve data for a single patient rather than to identify a population of patients with common characteristics or to facilitate the management of a specific clinical department. Typical data types which are often found within a CDR include: clinical laboratory test results, patient demographics, pharmacy information, radiology reports and images, pathology reports, hospital admission, discharge and transfer dates, ICD-9 codes, discharge summaries, and progress notes" [5, 6].

Thus, some changes have occurred in health activities that will generate an advanced culture in clinical issues, new roles will appear in health organizations. Functional, reliable, fully validated databases containing structured and unstructured data are the minimum requirement for the successful operation of the healthcare system.

The main objectives for the implementation of a secure medical data system must include a data warehouse that would integrate a set of specialized registers with data from source systems in real-time and able to process data flexibly, taking into account its heterogeneous nature. The data collected in the process of investigations and treatment are incorporated into the information system of the institution for internal use and are transmitted to the repository, according to the adopted regulation, for use in any medical institution to which the patient applies.

Among the issues that require resolution at the region or state level (in the case of smaller countries) is the degree of centralization of the repository: will it remain decentralized (distributed) in the units where data was created for each institution that is part of the national repository, or all databases and specialized registers are concentrated in the national repository. Many practitioners observe that a universal solution, in some cases, can cause more problems than it promises to solve.

The solution is not obvious and in each particular case depends on the legislation, as well as, to a large extent, on the administrative decisions, which should result from the presence of qualified personnel in each medical unit and the available financial means.

Another issue is selecting which personal data from the EMR will remain in the patient's EHR. For example, the results of the admission and discharge analyses of the hospitalized patient would be sufficient, and only in specific cases would it be necessary to include some from the course of treatment. Domain expert physicians must decide which components will move from the EMR to the EHR and into the repository for use by any other healthcare actors to which the patient applies.

The purpose of the data warehouse is to provide each category of actors in the health system with the necessary truthful data (safe, current, standardized) according to their own needs:

- *clinical doctors* – with the current data and the patient's complete treatment history included in the medical records, evidence-based treatment solutions, alerting and informing the doctor about the progress of the planned treatment, avoiding errors, giving the patient the necessary time;

- *hospital (clinic) managers* – with data on staff, stock of medicines, patients under care;

- *managers of medical analysis laboratories* – with the flow of patients requesting each category of analysis;

- *the National House of Health* – data on insured citizens, medical sick leaves for institutions and enterprises depending on the nature of the illness and the patient's occupation;

- *territorial authorities (ministry)* – data necessary for management at the respective level;

- *researchers* – anonymized data.

The access of the actors listed above as well as others (if necessary) must be performed in a sanctioned and secure manner.

The repository must have tools that would monitor the status and history of all data processing and would ensure security, integrity, finding and accessing data, re-use, and accessibility for the term set by specialists for each category of users. It is important that the repository is patient-centered so that it can be used as a reliable source of truthful data for any medical institution that the patient may later apply in the process of treatment or recovery. This will save patients and the health system time, effort, and expenses. On the other hand, the repository has a special role in the interaction between the treatment and care system and the health insurance system, an interaction that ensures the correct financing of expenses, the monitoring of the correct application of medication, the payment of allowances for the period of incapacity for work of employees [7, 8, 9].

Ensuring the security and confidentiality of information is very important. Since for each unit in the medical system, ensuring the protection of medical data by dispersed services is too expensive, a common solution for the entire system would be more effective and safer. Even advanced countries in the application of IT systems in health are not exempt from the problems that arise due to security deficiencies.

An example is on 10 June 2024, when seven hospitals in London were subjected to a cyber attack, as a result of which they had to cancel or redirect non-urgent operations, tests, and procedures to other institutions in the public health system, and employees switched to working with paper records. Hospital officials estimated that it would take several months to remedy the situation. More recently, the July 2024 attacks on 17 US hospitals and other US healthcare providers disrupted patient treatment, but no other data has been announced.

In any cyber attack, loss of database integrity, or disturbances of normal operation, detecting the causes and finding solutions to protect against further attacks, and avoiding the loss of data from the data warehouse are imperative both for the operation of the healthcare system and for maintaining the trust of users (clinicians, hospital managers, patients, etc.) in the usefulness of information systems.

4 Development, completion, monitoring of data in medical IT systems

In recent years, digital technologies have continued to transform the economy and society, people's activities, and daily lives. At the heart of this transformation are data-driven innovations.

In a society where increasing amounts of data are generated, the way data is collected and used must meet certain rigors and data protection rules. In [10], it is stated that "The aim is to create a single European data space – a genuine single market for data, open to data from across the world – where personal as well as non-personal data, including sensitive business data, are secure and businesses also have easy access to an almost infinite amount of high-quality industrial data, boosting growth and creating value, while minimizing the human carbon and environmental footprint".

Europe aims to reap the benefits of better use of data, including increased productivity in all areas, competitive markets, transparent governance, convenient public services, and improved health and the environment. In [11], it is stated that "Personalised medicine will better respond to the patients' needs by enabling doctors to take data-enabled decisions. This will make it possible to tailor the right therapeutic strategy to the needs of the right person at the right time, and/or to determine the predisposition to disease and/or to deliver timely and targeted prevention."

Current health regulatory models are based on access to individual patient health data and expanding its use and reuse so that health authorities have a secure basis for decisions aimed at increasing the system's sustainability and efficiency. Citizens will have the right to access and verify their personal health data and request their portability, an aspect implemented only fragmentarily and not yet ensured by laws and regulations.

The progress of the implementation of information systems depends on the commitment of healthcare actors to ensure the use of data following the General Data Protection Regulation (GDPR), according to which health data deserves specific protection. Although the GDPR has created fair conditions for the use of personal health data, the governance models for data access are different in different European countries.

Digital health services in different EU countries remain fragmented, although the European Commission has taken steps to strengthen citizens' access to health data and its portability. Article 40 of REGULATION (EU) 2016/679 OF THE EUROPEAN PARLIAMENT AND OF THE COUNCIL of April 27, 2016, on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing Directive 95/ 46/EC (General Data Protection Regulation) specifies that "In order for processing to be lawful, personal data should be processed on the basis of the consent of the data subject concerned or some other legitimate basis, laid down by law, either in this Regulation or in other Union or Member State law as referred to in this Regulation, ..." [12]. It establishes the Code of Conduct for the processing of personal data in the health sector. It is specified that any data transmission "may not be used for other purposes than those for which they were collected".

EHR development and health data interoperability are encouraged through the application of an Electronic Interchange Format of medical records. It facilitates the exchange of EMRs, medical images, patient laboratory results and electronic prescriptions between 22 EU states participating in The eHealth Digital Service Infrastructure (eHDSI).

The situation in the Republic of Moldova is a little different. Let's examine the example of the FORUM Clinic hospital. "Following the provisions of Law no. 133/2011 regarding the protection of personal data of the Republic of Moldova, the FORUM CLINIC hospital had the obligation to process personal data in safe conditions, for the general purposes specified in the context of providing the medical services you benefit from" [13]. It is noted that there is no automated decision-making process (including profiling) based on patients' personal data. FORUM CLINIC assumes responsibility and ensures security only for those information, personal data, and special categories of personal data (health data; genetic data; if applicable, biometric data), which are provided and processed by their staff exclusively and through the combination of the following means: "in the security perimeter of FORUM CLINIC; in IT and/or electronic systems/tools/devices, hardware and software, clinic records and forms; within the work schedule of FORUM CLINIC". The clinic was not responsible for the processing carried out in any other form, apart from that approved by them [13].

The situation changes through Government Decision No. 282 of 04-27-2022 "Regarding the liquidation of the Registry for Evidence of personal data operators", but there is no clarity on the structure of the medical data registers and the responsibilities for their formation and maintenance (https://particip.gov.md/ru/document/stages/cu-privire-la-lichidarea-registrului-de-evidenta-al-operatorilor-de-date-cu-caracter-personal-si-abrogarea-unor-hotarari-ale-guvernului/9006).

5 Medical IT systems in the Republic of Moldova and some sustainability problems

The state informational resource of the Republic of Moldova that contains information about patients including provided medical services, prescriptions prescribed by doctors, administered medicines, etc. is the Medical Registry approved by the RM Government Decision no. 586 from July 24, 2017. The Registry includes automated information systems: Primary Medical Care, Hospital Medical Care, Blood Service, Transplant Agency "Transplant", Human Resources Record System in the health system, and Register of evidence of vaccination against COVID-19 from 01.05.2021.

Each information system in the Medical Registry represents a separate record system, which is implemented and put into operation by its holder.

Thus, the National Medical Insurance Company is responsible and filling in the data on primary medical care, hospital care, and the portal of medical leave certificates.

The decision establishes that the Registrars of the Medical Registry are: the Ministry of Health, Labor, and Social Protection; medical and pharmaceutical service providers; and other institutions in the field of health care.

Data recipients are the Ministry of Health, Labor, and Social Protection; National Agency for Public Health; National Medical Insurance Company; medical service providers; health subdivisions of local and central public administration authorities; Public Institution "Information Technology and Cyber Security Service"; the General Inspectorate of the Border Police.

It is stated that the owner of the Medical Registry is the state, the holder of the Medical Registry is the Ministry of Health, and the keepers of the information systems that form the Medical Registry are the National Medical Insurance Company, the National Agency for Public Health, the National Blood Transfusion Center, and the Transplant Agency.

From this enumeration, it is not possible to establish the procedures by which the data from the patients' medical files are collected and how they arrive in the component systems of the Medical Registry. The logic of the components of this Registry does not stimulate the structure of an information system, which to a certain extent would establish similarities with the advanced health systems implemented in Europe or South Korea.

In Moldova, there are several implementations of medical data and images from different sources, and many EMR systems are installed in both the private and public medical sectors. An example is the DICOM network (http://dicom.md/Account/Login?ReturnUrl=%2f), a distributed medical image preprocessing and archiving system. The DICOM network was launched in Moldova in 2012 with the aim of providing access to collected imaging data for medical staff with access rights and also access for patients when they need personal data. Today, the system is implemented in several hospitals in Moldova; it collects and processes more than 5 TB of data per month with different types of medical equipment [14]. Most of these systems contain both medical data for the record of patients registered at these medical institutions and different types of medical image collections. The patient's personal data is the most sensitive and important information that should meet the main dimensions of data quality (completeness, correctness, timeliness, accessibility, compliance, etc.).

In the medical institutions of the Republic of Moldova, personal data are protected by the regulation of the National Center for the Protection of Personal Data of the Republic of Moldova, which is based on national legislation. In most hospital institutions, the internal standards or standard operating procedures for maintaining (https://scr.md/page/ro-proceduri-operaionale-269), the data are established from the patient record.

It is known that medical information systems can be secure and efficient, but they are highly dependent on the primary data based on which they operate and which contribute to clinical decision-making, planning, surveillance, and improvement of the health system.

Medical data includes both structured data from electronic medical records (EMR, EHR) and less structured data: ultrasonographic, radiographic images, etc.

An important problem for the quality of the collected data is the qualification and attitude of the personnel involved in entering data into the databases that are formed in the offices of family doctors, diagnostic laboratories, etc.

In many cases, in the tendency to report the implementation of IT systems, applications are proposed that have not been sufficiently tested and staff training has not been carried out in such a way as to be clear about the purpose and the method and to obtain professional skills.

These facts do not contribute to the popularization of computer systems – on the contrary, they harm the trust of users.

Perhaps the most requested, but also the one that often awakens patients' dissatisfaction, is the doctor's appointment.

Among the causes of user dissatisfaction are the interaction between the doctor and the IT system, through which the data on the doctor's activity periods are not transmitted in time, the reliability of the network and the application, and the discomfort for the inexperienced user.

We will analyze the example of the implementation of an application, which has a future – the prescription of compensated drugs by doctors.

Initially, the transition to this service throughout the country aroused the dissatisfaction of doctors, who were not sufficiently familiar with the specifics of the application (with a possibly not very successful structure and interface), of patients who waited too long until the doctor was able to access the system and to find out that, for each prescribed drug, it is necessary to enter the patient's INDP (13 digits) every time, but also of the pharmacists, who were not sufficiently trained. With time, however, the difficulties diminished, and the medical staff acquired the necessary skills; it is proof that the application is currently used and appreciated, and yet the National Medical Insurance Company (NMIC) from the Republic of Moldova received 1,450 petitions in the first six months of 2024 with regard the difficulties in registering at the family doctor, insurance with compensated medicines or medical devices, accessing the portal of medical certificates, etc.

Officially, there are no opponents of the implementation of IT systems in medicine, but the experience of many countries shows that the process is too slow and there is a gap between acceptance and use that can be attributed to various bottlenecks at different levels of healthcare in the system, primarily the level of training of staff but also age.

There are different causes, and in each institution, specific moments can be found not only for the healthcare system in Moldova but also for other countries, namely: resistance to changing the way, the style, the appearance of new duties in data collection, in some cases the medical staff who used to make the entries in the paper forms, now, in addition, it also fills in the electronic form.

It is important that the information system is developed primarily based on the requirements and expectations of the end users and necessarily by actively working with the developers.

6 Conclusions

Healthcare IT systems have historically been built fragmentary for different needs; they use diverse technologies and systems that were not designed to communicate with each other, and inefficient data exchange hinders data integration and sharing. The lack of interoperability leads to duplication of efforts and hinders the creation of a unified patient record. These shortcomings are specific not only to the Republic of Moldova.

As more and more data are accumulated in IT systems and different systems become more connected and more vulnerable to cyber attacks, protecting patient data and ensuring confidentiality become paramount and critical, or its deficiencies can erode trust and disrupt services.

Robust security measures, regular audits and staff training on cyber security are required, as well as the adoption of standardized, encrypted protocols and interfaces.

Throughout Europe and other parts of the world, the effective use of ICT in the health sector has been less successful than in industry, services, and other sectors. Despite aspirations, it has been estimated that up to 50 percent of IT systems projects using EMR have failed [8, 15].

The experience of many countries demonstrates that the success of implementation depends on both the internal components (structure, capacities, and available resources) and the external ones (social, economic, and political environment) in which the medical system operates [16].

Depending on the strategic vision and resources (financial and time) the system can be designed based on the global goal, the generally accepted standards of interoperability, or the incorporation into the system of the applications that are already working by adapting them to ensure the exchange of data in the entire system.

It would be natural for changes to be carried out through an iterative, continuous process at organizational levels according to a program agreed with all actors.

Technology alone is not enough to guarantee that potential benefits (efficiency and quality of care) can be achieved if service planning and organizational process management do not motivate staff to change and adopt.

User involvement is important to ensure compliance with work processes, but the design of medical IT systems also requires an overall systemic vision to ensure the interoperability of components, and the implementation to ensure operation according to society's expectations.

South Korea's launch of a new electronic health record system is state-led and offers patients greater accessibility and control over their data, although security and privacy issues are still not fully resolved [17].

The adoption rate of electronic record systems in medical institutions in Korea is more than 90%, and the government has a goal of complete digital transformation in healthcare. It is worth emphasizing that compatibility issues between institutions are resolved, and all EHR personal health records are stored in the single application My HealthWay, which is part of a wider program MyData, where all citizens at birth are registered with an individual identifier for any medical, financial, and administrative activity in Korea through a single registration.

The solution we propose for the Republic of Moldova is to develop a tool similar to My Personal Health Bank, available in a web application to help citizens obtain and be able to verify data related to healthcare, including treatments, diagnoses, tests, medications, allergies, etc. but also any services provided by the government and local authorities.

The healthcare field is being improved through the implementation of new treatment technologies but also through new ways of collecting, storing, and using medical data and information. Providing the entire population with high-quality services is paramount, and the resources available to the health system in any country are limited. A possibility to benefit from data necessary for treatment, research, and management is offered by medical technologies EMR, EHR, and DW (Data Repositories) that already ensure the exchange of data between different medical units, although still limited in some territories.

A gratifying and promising example is the collaboration between Denmark, Finland, Iceland, Norway, Sweden, the Faroe Islands, Greenland, and Aland in the exchange of medical data and the joint use of DW, reducing costs for each country's purposes (management, research) [16].

In the Republic of Moldova, there is a small number of medical data registers, a fact that would make it easier to solve the problem of creating a national repository, with an operating mechanism similar to My HealthWay, but the current problems, taking into account the state of affairs in the health system and the country, would be easier to overcome by calling for a collaboration with some European countries in the style of the Nordic countries.

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The Role of Artificial Intelligence in Inclusive Education

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Abstract

Inclusion is a process that can practically infinitely increase the level of learning and participation for all students. In fact, inclusion occurs simultaneously with the increase in the degree of participation. An inclusive school is a school in motion [1]. At the level of intention, the Government has demonstrated that it is ready to accept children with special educational needs in schools in the community so that the children can return to their families, but at the level of legislative regulation, much remains to be done. The teachers do not know the methodology of working with these children, they do not know how to grade them, what kind of tests to apply at the end of the school year and if they do, is it legal? Many unanswered questions at the moment, but we hope that in the near future we will find answers to many of them. One of the methods of helping teachers and parents in the education and development of children would be Artificial Intelligence, which would allow working with each child individually and at their own pace.

Keywords: Inclusion, Artificial Intelligence,

1 Introduction

Poverty, migration, dispersal of families, disability are just some of the factors that determine the risk of social isolation of some categories of individuals, the principles of human cohesion and solidarity being strongly affected. The most obvious manifestation of this phenomenon is exclusion from education, with strong repercussions on social integration in general. That situation explains the special importance given to inclusive education as a phenomenon and process promoted in order to establish social equity. Promoting inclusion and teaching from an inclusive perspective requires a broad vision and specific skills that all teachers must possess, not just, as is mistakenly believed, those who directly assist a child with special educational needs. Inclusive education and education systems in general have evolved to the point where all teachers need to know that diversity is present in groups of children at all levels of education and that addressing the different requirements of children is what fundamentally marks the new trends in education. The application of Artificial Intelligence could come as a help in organizing the educational process. Creating platforms that would allow the process to be more convenient for each child depending on their requirements. Some children need a slower pace, others a faster tempo.

2 The benefits of artificial intelligence in education

Artificial Intelligence (AI), which is being talked about more and more, brings many significant benefits in the field of education, improving the way we learn and teach. AI can adapt the learning process to the individual needs of each student. By collecting and analyzing data about each student's performance, preferences, and learning style, AI can deliver personalized learning resources and materials. This personalization helps to effectively optimize learning and increase student academic success. AI systems can serve as virtual tutorials, providing interactive explanations and demonstrations to help students understand complex concepts. Can create tests and assessments that adjust to individual children's knowledge and abilities. It can reduce the administrative burdens of teachers and school staff by automating activities such as grading tests, checking plagiarism, scheduling classes, or managing student data, which allows the teachers to focus more on teaching. AI is able to find and eliminate mistakes in educational materials. AI contributes to the creation of engaging and interactive content, the creation of interactive and educational games. AI enables distance learning for students who live in more distant areas. Artificial

Intelligence brings a diverse set of benefits to education, from personalizing learning to improving accessibility and continuous innovation. However, it is essential to maintain a balance between the use of technology and the direct involvement of teachers and school staff to ensure a complete and successful educational experience for each student [2].

3 The challenges of using artificial intelligence in education

Apart from the benefits of AI in education, there are also challenges that need to be addressed. One of these challenges is related to the fact that AI can generate racial and gender stereotypes. There may also be issues related to the protection of student data and how this data is collected, stored, and used.

Other important downsides that can arise with the use of AI in education are:

1. **High costs:** Implementing AI in education can be expensive and require significant investment in infrastructure and equipment.

2. **Required skills:** Using AI in education may require advanced technical skills, which may be limited among teachers and educational administrators.

3. **Dependence on Technology:** Excessive dependence on technology can affect students' communication and social interaction skills, which can have negative consequences on their personal development.

4. **Perspective Limiting:** The use of AI can limit students' perspectives by presenting content and activities that reflect their prior interests and preferences without exposing them to new ideas or perspectives.

5. **Ethical responsibility:** The use of AI can raise important ethical questions regarding the accountability and transparency of decisions made by algorithms and how they may affect the lives of students [3].

4 Conclusion

Artificial intelligence was founded as an academic discipline in 1956, and since then it has gone through many transformations. It represents the intelligence exhibited by machines as distinct from the natural one exhibited by humans. However, it remains in constant development and applied in different fields. One of the very important areas is the knowledge base established since school. Every year the children's promotions differ and not always the classic teaching methods are successful. Children get bored more quickly, easily evade. Fun and educational methods must be found constantly so that the student does not lose interest in learning. At the same time, the number of children with special needs is growing rapidly in all regions. And the teachers are not prepared for inclusion.

One of the options is the preparation of new frameworks and platforms that would help both teachers and children to understand and use the knowledge received from school in the future. Artificial Intelligence could be a solution to help not only children but also teachers.

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Credibility of Textual Information Generated by AI Tools

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Abstract

The paper refers to the topic of information credibility in the context of the increasingly active use of AI tools. Starting from the notion of the credibility of the information, the examples of AI tools that generate credible text content are presented. At the same time, the particularities of generating credible textual content with the help of AI tools were discussed, as well as the way we can detect texts produced by AI tools.

Keywords: credibility, AI tools, content generation.

1 Introduction

Now more than ever it is very easy to disseminate and make information known. Social networks have become the perfect means to provide speed to become viral on Web with information that must be known by a large number of users. Moreover, wanting to find credible information, we can encounter the phenomenon of misinformation (unintentional inaccuracies) and/or disinformation (deliberately deceptive information).

False information makes it difficult to make the right decisions. That's why when it comes to misinformation about a company, it's about large amounts of money that can be lost. In other cases, when it comes to influencing political opinions, it can influence the results of elections in a state.

On the other hand, the identification of credible information is an important thing, given that certain decisions that can save human lives depend on the information found on the Web. A relevant situation is the COVID 19 crisis when truthful information about vaccines was much needed.

A current trend that seems to be more and more developed refers to the fact that if before Web content was created exclusively by humans, now we are increasingly talking about content generated by artificial intelligence tools. The distinction between human-generated content and AI-generated content is often complicated.

Therefore, in the conditions of an abundance and excessive accessibility of information, it is important to identify means that could help us to be able to choose credible information that is suitable for safe use.

We will try to answer this purpose to some extent in this article. Starting from the notion of the credibility of the information, the examples of AI tools that generate credible text content are presented. At the same time, the particularities of generating credible textual content with the help of AI tools were discussed, as well as the way we can detect texts produced by AI tools.

2 The particularities of credible information

Daily we consume information to plan our activities, make certain decisions regarding different personal areas that may be related to health, finances, education or politics. Credible information helps us make rational decisions based on reliable facts avoiding assumptions. Incorrect medical advice could lead not only to a simple incorrect decision, but may even cost human lives. By ensuring the credibility of the information, we can minimize the risk and prevent harm caused by erroneous decisions.

It is understandable that if we cannot know the source of the information, then the information is less credible. At the same time, information with high credibility comes from the government, mass media, educational systems, or scientific communities. If information from trusted institutions appears less certain, then trust also erodes and skepticism arises in the information delivered by such institutions. Therefore, ensuring the credibility of the information protects people from manipulations. Credible information is a tool to hold society accountable against possible moral vices such as corruption, lies, misinformation, manipulation, etc. In the educational, scientific, and technological field, credible data is a catalyst for progress, given that it represents the foundation built by previous generations of information obtained and verified, thus being credible sources for further research.

Information credibility can be defined as people's assessment of the degree to which they believe in the quality of the information content [1]. Respectively, it can be assumed that credible information is that which can be trusted, up-to-date, accurate, and transmitted by verified sources or competent experts.

The key qualities of the credible information confirmed by our experience and [2] are: accurate information based on objective facts without any personal opinions; trusted source usually with a reputation built over time and with a concrete area of expertise; clear existence of the origins of the information starting from the author, the source, and the way of collecting the information; evidence that confirms good documentation before the information is collected; an updated information to the concrete context at a given time; without any personal, political or financial opinions; the continuation of the logical chain of other information already known previously and which already is approved as true; a well organized so that it is clear. Credible information in general would be that which can be trusted because it has been analyzed, studied, is clear, and does not express anyone's opinions and interests, thus being a safe basis for decision-making and forming opinions.

3 AI tools in content generation

IT tools that generate texts are only tools that can assist in the creation of content according to certain initial settings. What AI tools can do is the closest thing to a human-model content creation mechanism based on accumulated experience. That's why quality depends on how it's used and the data it's trained on. Some example of AI tools and technologies designed to aid in generating or verifying credible information are presented in Table 1, separated in 6 groups.

Table 1. AI tools and technologies designed to aid in generating or verifying credible information

AI-Based Fact-Checking Tools	Content Generation and Research	Automated Research Assistants	Misinformation Detection	Source Evaluation and Analysis	Natural Language Processing (NLP) for Information Extraction
Google Fact Check Tools	OpenAI's GPT Models	Elicit	Deeptrace	News Guard	Diffbot
Claim Buster	IBM Watson Discovery	Semantic Scholar	Hoaxy	Ad-Verif.ai	

AI is capable of generating credible information content summaries from credible information [3]. Even when we are not sure about the credibility of the information, AI can help cross-referencing check the veracity of the information, thus comparing the new information with the already existing and previously verified information. Being based on all grammatical linguistic rules, AI tools, being generated on the basis of credible texts and verified by peer review, will generate credible textual information.

AI tools by checking previous sources of credible information is able to check information for unusual or well-determined message to disinformation. AI can generate uncredible information if there are the following moments. The output generated depends on the quality of the input data to AI tools. Likewise, the context of the generated content may not reflect the context of the input data from which the new information was generated. Therefore, in order to misinform, AI tools, being used with wrong data, can generate content that will require checks to avoid intentional misinformation. That's why human verification is crucial to avoid some weird results from AI tools when generating content. One of the special points of AI is the non-transparency of the generation of results; therefore, it is necessary to know the sources and methodology of the generation of AI tools. Thus, ethical guidelines must be developed to prevent abusive misinformation of AI tools.

4 Conclusion

The subject of the credibility of information in the age of the Internet is a permanent current one, especially since this information can be generated with AI means. In such conditions, we face not only one method of text generation, such as creation by humans, but also automatic generation by specialized AI tools.

This topic is all the more interesting because it raises the question of whether texts generated by AI tools could be considered credible by software tools designed to evaluate human texts.

It is also important that the possibilities of generating credible textual content are available not only for English but also for other languages. However, we note that not all tools have such options.

However, we note that the value of AI tools for now is valuable together with the supervision and critical thinking of the people who consume the information, thus guaranteeing the credibility of the contents.

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TransShare: A Peer-to-Peer Car Sharing Platform

Amalia-Nicola Harton, Cosmin Irimia, Madalin Matei

Abstract

This paper presents the implementation details and technological framework of "TransShare", a distributed web application designed to facilitate the temporary sharing of private vehicles between users. The platform supports both the listing and rental of cars within the territory of Romania. The implementation has been reimagined using a peer-to-peer (P2P) mechanism to decentralize operations, improving system efficiency, scalability, and reducing reliance on central servers. Additionally, the platform integrates OpenAI for enhanced interaction but introduces an optimization layer to mitigate potential operational costs associated with AI queries. The system was developed using modern web technologies, namely the MERN (MongoDB, Express.js, React, and Node.js) stack, and adheres to the Model-View-Controller (MVC) architecture to ensure scalability, maintainability, and modularity.

Keywords: Peer-to-peer car sharing, AI-driven optimization, cost-efficient operations, MERN stack architecture.

1 Introduction

The rise of shared mobility solutions and decentralized architectures has revolutionized how users access transportation. "TransShare" seeks to capitalize on this trend by offering a web-based peer-to-peer (P2P) [1] system for car sharing. The platform allows private car owners to list their vehicles for temporary rental, while users seeking cars can browse and filter based on various criteria. This approach fosters a more decentralized and efficient method for sharing resources, reducing dependency on central server infrastructures and increasing scalability.

Moreover, the platform leverages OpenAI [6] to provide users with real-time assistance. However, recognizing the cost implications of frequent interactions with AI, we introduce an additional optimization layer to manage and minimize the operational costs related to OpenAI queries, ensuring a balance between AI functionality and affordability.

2 Proposed system

Traditional client-server models in web applications often face limitations in scalability and centralization of control. In contrast, peer-to-peer (P2P) systems decentralize operations [5], enabling direct interactions between users. This paradigm shift has significant implications for platforms such as "TransShare".

2.1 Peer to peer model

The P2P model within "TransShare" allows users (both renters and car owners) to communicate and exchange information directly, bypassing the need for a central server to manage all transactions. This is achieved by employing distributed technologies where each user acts as both a client and a server, managing their data locally while sharing only the necessary information for transaction processing.

The P2P system in "TransShare" functions by allowing users to upload, store, and access vehicle data locally. A distributed ledger or decentralized database can be used to maintain records of all transactions and agreements. Once a car rental request is accepted, direct peer-to-peer communication is established through the platform's secure messaging service, allowing the renter and owner to finalize logistics without the need for continuous server mediation.

2.2 Cost optimizations

A major feature of "TransShare" is the integration of a virtual assistant powered by OpenAI, designed to provide real-time assistance to users. This AI-driven feature can answer inquiries about vehicle details, rental processes, and other relevant information, significantly improving the user experience.

However, the frequent use of OpenAI-based queries can lead to high operational costs [3]. To address this, an optimization layer is introduced to manage and reduce the frequency of unnecessary AI interactions while maintaining responsiveness.

The optimization layer functions by filtering user queries through a tiered system:

- Local Cache of Frequent Queries: Commonly asked questions and standard responses are cached locally. The system retrieves answers from this local database before making a call to the OpenAI service, thereby reducing the need for frequent API requests.

- Contextual Understanding: The system leverages natural language processing (NLP) to determine whether a query requires a fresh response from OpenAI or can be answered using previous interactions. This approach minimizes redundant calls to the API.

- Usage Monitoring and Rate Limiting: The platform monitors AI query usage patterns and can enforce rate-limiting measures to avoid excessive API calls during periods of heavy usage.

- Batching and Delayed Responses: Non-urgent queries can be batched and processed during off-peak hours or combined into a single OpenAI request, reducing the overall number of API interactions.

By implementing this optimization layer, "TransShare" ensures that users benefit from the real-time intelligence provided by OpenAI without incurring prohibitive costs. This solution balances the need for advanced AI-driven features with the economic realities of maintaining a cost-effective service.

The system leverages the MERN stack [4] (MongoDB, Express.js, React, and Node.js), which supports both the P2P architecture and the integration of OpenAI. MongoDB, a NoSQL database, serves as the decentralized database for storing vehicle listings and transaction histories, ensuring scalability and efficient data retrieval.

The application's frontend, developed using React, provides an intuitive user interface for listing and searching vehicles. Node.js serves as the runtime environment for the server-side code, enabling the seamless handling of network requests. Express.js manages the server infrastructure and facilitates secure communication between users in the P2P environment.

2.3 Features and Functionality

"TransShare" provides a comprehensive set of features for both car owners and renters. Users listing a vehicle can input detailed specifications, accompanied by high-quality images, while renters can search for vehicles based on a wide array of criteria such as location, model, price, and date. Once a rental request is accepted, secure direct messaging is facilitated through the P2P network, allowing users to finalize logistics without requiring continuous server-side management.

Additionally, OpenAI assists users by answering questions related to the platform, enhancing overall usability. The introduction of an optimization layer ensures that the costs associated with this AI interaction are kept in check.

3 Conclusions

This paper has outlined the implementation of "TransShare," a distributed P2P web application designed to facilitate car sharing in Romania. The transition to a peer-to-peer architecture enhances scalability, efficiency, and reliability, while the integration of OpenAI, complemented by an optimization layer, provides intelligent real-time interaction at a manageable cost. Through the use of the MERN stack and adherence to the MVC architecture, "TransShare" achieves a balance between performance, scalability, and user experience, establishing itself as a forward-thinking solution for peer-to-peer car sharing.

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New Rules for a Newer and More Challenging Game of Chess

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Abstract

The goal of this project from the beginning was to create a mobile application that allows users to play chess more interactively. The application is aimed at chess enthusiasts (amateurs or professionals) who study it in more detail and who play regularly. Currently, many users play chess exclusively online, either on a web or mobile application, while lessons and study are conducted through courses or free online engines. In this context, the proposed application extends the standard formats of the chess game into a variant with features that are currently not found in a standard game.

Keywords: chess, web application, interactive game.

1 Introduction

Chess is one of the world's oldest and most complex strategy games, with a history spanning over a millennium [1, 2]. Originating from India, where it was known as Chaturanga, the game spread to Persia, then the Islamic world, and eventually to Europe, becoming popular worldwide [1]. Interestingly, it also has a four-player variant (see Figure 1 on the right). Over the centuries, chess has been appreciated not only as a form of entertainment but also as a means of cultivating intellectual and strategic skills [3].

The chess set consists of a square board divided into 64 squares, alternating in light and dark colors, and 32 pieces equally divided between two players. Each player has a king, a queen, two rooks, two

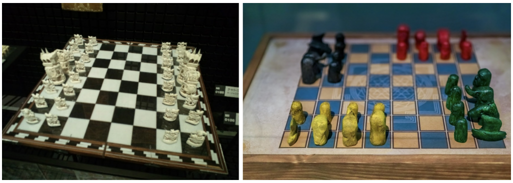

Figure 1. Example from Chaturanga from Rajasthan, India (two players on the left and four players on the right)

knights, two bishops, and eight pawns. Each type of piece has a specific way of moving, and the main goal of the game is to checkmate the opponent's king, that is, to place it in a position where it cannot avoid being captured. Chess is not only a test of individual skills but also a symbol of strategic thinking and long-term planning. Players must anticipate their opponent's moves, protect their pieces, and create opportunities for attack. This requires not only a deep knowledge of the rules and tactics but also the ability to think several moves ahead, evaluate positions on the board and make quick and precise decisions.

In the modern era, chess has experienced a revival due to technology. Online platforms allow players of all levels to face opponents from around the world, analyze famous games, and improve their skills through interactive training. Additionally, computers and artificial intelligence have brought new dimensions to chess analysis and understanding, offering unprecedented learning and competition opportunities (see the matches from 1985-1996 between the great Russian champion Garry Kasparov and IBM's Deep Blue [4, 5]. These developments have cemented chess as a timeless game that continues to fascinate and challenge curious minds everywhere [6, 7].

2 Architecture and Implementation

The application is structured using the MVC (Model View Controller) model. The data modeling and main functionalities will be presented below.

2.1 Data Modeling

Data is modeled using 8x8 matrices. In the initial stage of the project, we used standard chess notation (Forsyth Edwards Notation - FEN) to mark each piece on the board. FEN is a standard used to represent a specific position in a chess game, providing a simple and compact way to describe the arrangement of pieces on the board and other relevant game details [8].

2.2 New Rules and Moves

The pieces can receive upgrades during the game. Instead of creating rules that occasionally apply to each piece, we decided to isolate each upgraded piece and replace the standard ones when necessary.

UKNIGHT and UBISHOP* - These pieces don't exist specifically in the game, the upgrade in their case involves replacing one piece (knight or bishop) with the other, with specific situations favoring one over the other.

UPAWN - standard pawn rules are maintained but with the ability to capture diagonally and move two squares instead of one. The pawn can be promoted to any piece (QUEEN, ROOK, BISHOP, KNIGHT, UROOK, UQUEEN) when it reaches the final rank.

UROOK - standard rook rules are maintained, with the ability to jump over any piece during movement (without capturing it) to the next square, provided no piece of the same color occupies that square.

UQUEEN - reduced ability to move diagonally more than one square but otherwise maintains the standard queen rules. This upgrade is theoretically a downgrade, with the advantage being that the change is chosen by the player for their opponent, not themselves.

3 Usability Tests

We tested the application to discover similarities and differences between this and a standard chess game. The tests were conducted only in single-player mode, so we cannot provide a 100% accurate description of what to expect from the app in two-player mode. However, we can use my extensive chess experience, both online and over the board, as a useful tool in these comparisons.

Similarities: (1) The game format is similar, making it easy to understand the rules after a few turns for someone who knows standard chess moves, (2) Balancing thinking and analyzing the available moves with the time allocated for each move, maximizing one and ignoring the other gives an advantage to the opponent, (3) Classic chess principles apply to both: bring out all your pieces early in the game, protect the king, capture pieces advantageously, and create an attack to checkmate the opponent.

Differences: (1) Extra time is needed to manage upgrades advantageously, (2) Less time to find the best moves, (3) Lack of experience with these possible upgrades reduces the use of standard chess openings, as an upgrade can disrupt an entire attack plan, (4) The existence of upgrades favors aggressive play, as the pieces can quickly get close to the king, (5) A much higher probability of fatal mistakes due to the difficulty in accounting for specific upgrade moves.

4 Conclusion

The usability tests we performed showed the interest of chess players in the proposed version, demonstrating the potential of this project. After creating the Improved Chess application, several aspects related to the creation process and the game itself became evident. Here are the main takeaways: (1) Using a real device instead of an emulator to run and test the application is usually an advantage. Connecting an Android phone to Android Studio is easily done, whilst an emulator can consume a lot of resources on the host device, possibly hindering both debugging and compiling, as well as the app's actual running, which may not provide an accurate experience of the game. (2) Managing the application in an MVC format simplified the separation of its main components. Without such an approach, implementation could become confusing, making it harder to search, add, and change code.

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Automating Rhythm Game Level Design in Unity Through Beat Detection

Cosmin Irimia, Stefana Biceadă

Abstract

Technology is integral to our lives, impacting everything from daily tasks to critical fields like healthcare. While computers excel at technical tasks, translating intuitive concepts like rhythm into computer logic can be challenging. Detecting rhythm is natural for humans, as seen in activities like tapping or nodding to music. Over time, various beat detection algorithms have been developed for applications such as audio visualization and cardiology, where they monitor heartbeats and rhythms to identify diseases.

Keywords: Game development, sound processing, content generation.

1 Introduction

Rhythm games have gained popularity due to their skill-based gameplay, catchy songs, and engaging visuals. Well-known games in this genre offer large libraries of familiar songs, enhancing their appeal. These games require players to interact with events synchronized to the music's rhythm, with varying difficulty levels. While levels are hand-made, offering polished results, this method limits players to the game's song library and slows updates due to the time-consuming level development process.

Rock songs have simpler rhythms, while classical music presents more complex challenges. The goal of this project is to develop a rhythm game that allows users to upload any audio file to generate levels.

2 Existing methods

This section aims to provide the general state of the field at the current time, by presenting already existing methods, as well as their strengths and weaknesses.

2.1 Offline versus Online Beat Detection

There are two primary methods for analyzing audio information: offline and online beat tracking. Offline beat tracking uses a pre-existing database to analyze musical events, allowing for detailed examination and fine-tuning, as it can account for patterns and the overall intensity of the music. In contrast, online beat tracking analyzes real-time audio input frame by frame, which can lead to less reliable results because it only has access to past events and must make immediate decisions. While offline tracking offers greater accuracy, online beat tracking is more suited for live performances, where real-time analysis is required. Both methods have distinct advantages and use cases.

2.2 Onset detection

Onset refers to the moment a musical sound becomes perceptible, such as note changes or drum hits. Onset detection algorithms focus on identifying sudden changes in amplitude to detect these events. Early approaches in the mid-1990s focused on monophonic audio, where only one note is played at a time. Over time, these algorithms have evolved to handle polyphonic music, where multiple instruments play simultaneously. Onset detection algorithms typically involve three steps: preprocessing to highlight relevant information, reducing the input with a reduction function, and peak detection to identify onsets. Variations of these algorithms have become more sophisticated over the years, incorporating psychoacoustic [6] principles like frequency resolution and pitch perception to closely mimic human auditory behavior. Klapuri's [2] work, integrating psychoacoustics, led to significant advances in onset detection and won the ISMIR 2004 tempo induction contest.

2.3 Rhythm pattern recognition

Another research topic that is worth discussing about, although it did not receive as much attention as the other approaches that were considered, is rhythm pattern recognition. This refers to the patterns in the rhythm of a song that are recurring in the form of arrangements of beats, and accents. While in theory the concepts of rhythm pattern and tempo are related and could result in similar outputs, the first one received less attention, most probably due to the complexity of creating datasets. Various methods have been developed to recognize rhythm patterns in an audio file. For example, [3] proposed a beat histogram that was derived from an autocorrelation function. Among other notable papers there is [4] in which a harmonic analysis of the rhythmic patterns is proposed.

2.4 Threshold determination methods

Another important aspect when it comes to accurately determining musical events is the determination of the threshold values. Establishing appropriate values can contribute to the accuracy, therefore multiple approaches have been presented.

2.4.1 Fixed Thresholding

This approach involves experimenting with different values and levels and observing the results in order to manually fine-tune the values. A clear disadvantage is its non-adaptiveness to changes in amplitude, resulting in low performance for songs that vary dynamically. Nevertheless, implementing a fixed threshold is the simplest approach being very easy to understand, and it may find its use case in songs with consistent volume and predictable dynamics.

2.4.2 Dynamic Thresholding

Dynamic Threshold: This method involves adaptively adjusting threshold levels based on a previous set of data in order to keep a specific sensitivity. Although it is more difficult from an implementation point of view, it is more flexible and responsive to changes in the signal, resulting in better detection accuracy. Some things to take into consideration when implementing this method is the parameter tuning such as selecting the appropriate window size and scaling factors. Although dynamic thresholding can adapt to changes in signal amplitude, it may still be sensitive to noise. If the signal contains a lot of noise, the calculated thresholds might be effective, resulting in false detections.

2.4.3 Statistical Thresholding

This method utilizes the statistical aspects of an audio signal in order to determine appropriate values for the threshold. There are multiple techniques used in order to determine the values such as:

- Mean and Standard deviation: The threshold is set based on the computed mean using the amplitude, which reflects the overall intensity or energy of the signal at a given time. By analyzing the mean, we can understand better how a song's signal fluctuates

- Percentile-Based Thresholding: Unlike the later technique, percentile-based algorithms capture the behavior of the song's musical events distribution. This approach focuses on capturing the extreme values within the signal. This means that the lower values of the frequencies are being ignored and they do not contribute to the computation of the threshold itself, making it very effective against noise. One thing to take into consideration is the sensitivity, higher percentiles correspond to a more strict result, while lower percentiles give more lenient values, with a wider range of signal detection.

2.4.4 Machine Learning-Based Thresholding

Machine learning-based thresholding. These methods offer a great approach when it comes to complex musical event detection across a diverse dataset and have very practical implementations in important fields such as healthcare monitoring. Some of the most popular methods are Supervised Learning Methods, including Support Vector Machines and Convolutional Neural Networks [5] and Unsupervised Learning Methods such as clustering algorithms, as well as Reinforcement Learning Methods such as Q-Learning.

3 Project Implementation

Building upon the knowledge presented in the previous chapter, this chapter explores the methodologies and overall development of this project, including the audio analysis, the feature extraction and the generation of a rhythm game level.

3.1 Workflow overview

The main goal of the project is for each user to input an audio file of their choice in order for that to be analyzed and become a playable rhythm game level. Therefore the project begins with inputting a song/audio file provided by the user. This file can be of any compatible format (e.g., MP3) which will later be processed by the algorithm.

3.2 Song analysis

The song analysis algorithm, implemented in C# using Unity, processes audio files provided by users to generate game levels. Users can select any song in .mp3 or .wav format from their computer, and the file is attached to a Game Object that handles audio playback through Unity's AudioSource component. This component offers various features, including the 'GetSpectrumData' method, which retrieves the frequency spectrum data from the audio file. The method takes three parameters and plays a key role in analyzing the audio to create the corresponding game levels.

- samples: an array of floats where the spectrum data will be stored

- channel: the audio channel that is being used to extract the spectrum data, either 0 (for the left channel) or 1 (for the right channel)

- window: the type of window to use in order for the FFT to use. Taking into consideration the points made in the 6th Chapter of this paper, for this project, the windowing function that was chosen was BlackmanHarris because of its great reduction when it comes to spectral leaking. In order to make it as general as possible, the algorithm takes into consideration both channels.

The "Update" method in Unity is used to call 'MakeFrequencyBands' once per frame during the song, processing audio data by dividing it into 8 frequency bands. A logarithmic scale is applied, allowing lower frequencies to be captured in finer detail while still including higher frequencies. The frequency bands are normalized using the highest recorded value to ensure all values range between 0 and 1. Additional methods calculate amplitude and create buffers for smoother results. The processed data, along with timestamps and amplitude, is serialized into a JSON string and saved locally in the project's assets for future access and use.

3.3 JSON analysis and calculating the threshold

The data from each JSON is deserialized and processed to calculate threshold values, with frequency bands combined for clarity. Higher frequency bands are merged to improve algorithm efficiency. Several threshold computation methods were tested, with percentile-based thresholding chosen due to its effectiveness in handling noise and focusing on amplitude spikes. This method also allows for difficulty scaling by selecting predefined percentiles. To optimize computation, mini-batching is used, dividing data into smaller parts for more accurate results. Tailoring thresholds for each frequency band improves detection accuracy, enhancing system performance.

3.4 Generating obstacles

The rhythm game level itself now has all the components it needs in order to be generated. The same approach that was used when initially analyzing the songs serves as the base on the level generation as well. The specific audio Source is played and observed per frame, thanks to the Update method from Unity. The data from the JSON is again deserialized from the file. Within this method the script is continuously checking the current timestamp of the audio that is playing in the background and checks it with the timestamps from the saved file. As the audio reaches the timestamps found in the file, the scripts check for the band values and the threshold values and determine if game objects that serve as the obstacles need to be activated. Upon being activated, they are given a certain force in order to simulate the movement.

The mini batching is again employed in this part of the project's code, the script incorporating logic that determines exactly which of the threshold values to use for each of the determined batches.

4 Conclusion

Our work set out with the purpose of developing a rhythm game that diminishes the amount of manual work developers need to put in to create a level. This thesis included a developing part as well a theoretical research part covering techniques such as Fourier transforms, filtering and thresholding. By understanding the basics of signal processing and researching various algorithms, the project presents an algorithm with a strong foundation in the realm of beat detection [1]. By combining the knowledge acquired from this theoretical part with game design elements, this project accomplished its goal of creating a game that can take as input any audio file and generate a playable rhythm game level by analyzing the frequencies of the inputted audio file. In conclusion, while the current state of the project indeed accomplishes its primary goal of automating a manual process, it also allows for future improvement.

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ukuApp - An Android Learning Environment for Beginner Ukulele Players

Maria Nestor, Adrian Iftene

Abstract

ukuApp is an application made for helping beginners who study the ukulele, an application that has a tuner and a component for learning the basic chords. Moreover, it has an integrated feature where each user can contribute known songs along with their chords. The application is designed to minimize the number of applications needed for learning this instrument, by having integrated the most important parts for a beginner to start with.

Keywords: ukulele, Android, Learning.

1 Introduction

1.1 What are ukuleles?

A *ukulele* is a four-string musical instrument, similar to a guitar. The origin of this word comes from Hawaiian, and is translated to “jumping flea”. This instrument was brought to Hawaii by the Portuguese immigrants. The locals were fascinated by the rapid finger movements of the players and came up with the name “ukulele”, where “uku” translates to “flea” and “lele” to “jumping” [1].

Nowadays, ukulele’s popularity comes from famous songs such as *“Somewhere over the Rainbow”*[1], *“I’m Yours”*[2], *“Hey, Soul Sister*[3],

[1]Performed by Israel Kamakawiwo’ole - https://www.youtube.com/watch?v=V1bFr2SWP1I&ab_channel=MountainAppleCompanyInc

[2]Performed by Jason Mraz - https://www.youtube.com/watch?v=EkHTsc9PU2A&ab_channel=JasonMraz

[3]Performed by Train - https://www.youtube.com/watch?v=kVpv8-5XWOI&ab_channel=TrainVEVO

and from their exposure by known singers such as *Elvis Presley*, *Jake Shimabukuro*, or *Israel Kamakawiwo'ole*.

The ukulele is a member of the lute family of instruments, the same as the Mandolin, Banjo, or Cobza. It is made of wood or plastic, and it resembles a small acoustic guitar, which sets it apart from other instruments in its family. The exception is the pineapple ukulele, which has the shape of a lute. The ukulele has four strings and comes in different sizes. The common ukuleles are soprano (known as standard ukulele), concert, tenor, and baritone.

Figure 1. Most common types of ukulele[4]

1.2 Similar Applications

Ukulele Tuner protect[5] – This application serves as an essential tool for ukulele enthusiasts, offering a versatile tuner with a range of tuning options, such as D-Tuning, Low G, and Standard, among others. Beyond its tuning capabilities, it provides a comprehensive chord

[4]Image source: `https://mississaugafineartsacademy.com/ukulele-lessons-mississauga/`

[5]`https://play.google.com/store/apps/details?id=com.myrapps.ukuleletools`

library, catering to the needs of ukulele players at any skill level. Its user-friendly interface makes it not only simple to use but also indispensable for achieving pitch-perfect sound quality.

Pocket Ukulele[6] – This application has a large collection of ukulele chords, making it an invaluable resource for musicians of all levels. With its intuitive interface and user-friendly design, it caters to both beginners and advanced players alike. Featuring over 5000 ukulele chord diagrams, it serves as a comprehensive tool for learning new chords or refreshing one's memory of familiar ones. Its simplicity and effectiveness make it an indispensable companion for anyone looking to enhance their ukulele skills.

Ukulele Tabs[7] – This is the mobile application of the website Ukulele Tabs and Chords[8]. It stands as the most extensive repository of ukulele tabs and chords, frequented by thousands of players seeking musical inspiration. One of its standout features is its open platform, allowing any user to contribute to its vast collection of songs. This collaborative approach enables users to learn their favorite tunes while also offering a platform to share tabs and chords for songs not yet included in the collection. Through this interactive community, players can both enrich their musical repertoire and assist others in their musical journeys.

2 Application Architecture

The ukuApp application has 18 activities and 3 dialogs. They are structured into four main modules: tuner module, community module, lessons module, and test module. To reach them, the user meets the *Main Activity*. This activity contains a TextView having the title of the application (*ukuApp*) and four buttons to access, in order: (1) the tuner module, (2) the community module, (3) the lessons module,

[6] https://play.google.com/store/apps/details?id=com.ukulelechords
[7] https://play.google.com/store/apps/details?id=com.ukuleletabs
[8] https://www.ukulele-tabs.com/

and (4) the test module (see in Figures 2 and 3 the main and tuner activities).

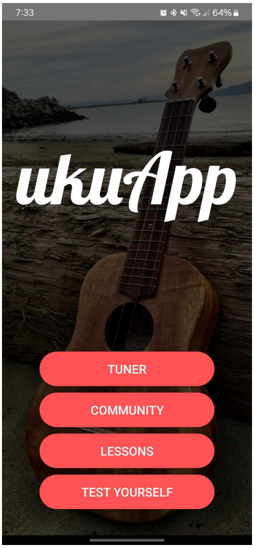

Figure 2. Main Activity

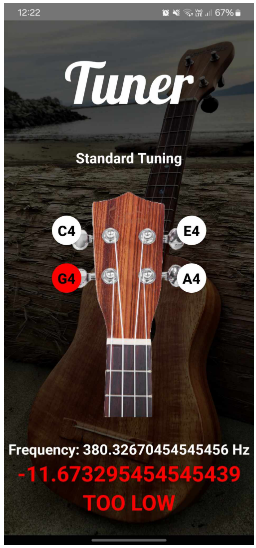

Figure 3. Tuner Activity

The DiagramDialog is made to show the image resource of the chord (see Figure 4). The user can see how their fingers should be positioned on the ukulele and can then play the chord. The InfoDialog is created to explain the CardViews' functions for the users to understand what every button does (see Figure 5).

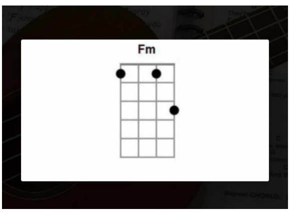

Figure 4. Diagram Dialog

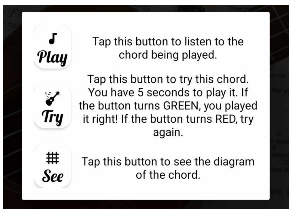

Figure 5. Info Dialog

For testing this application, a Google Form was created. The role of this form is to collect suggestions from real users, to make this application better and more user-friendly. This form has multiple-choice questions (mostly) and open questions. This form was applied to persons having experience in ukulele and/or guitar or no experience at all. In this testing process, five persons participated, between 18 and 44 years old, two males and three females. 20% of the respondents have already some experience, and the rest of 80% have no experience at all. They appreciated the applications and come with two interesting suggestions: (1) in the Lessons Module, an indicator for when the mic is working; (2) in the Community Module, for each song to add the chords used. Overall, the application has fulfilled its mission. Its functionalities were helpful for beginners (based on their reviews and the reviews coming from those with experience). The application is intuitive and useful for the target users.

3 Conclusions

The *ukuApp* application gives the possibility for anyone to learn the ukulele without a professor. It brings together three important areas in the learning process so that the user has everything they need at their fingertips for their journey. The user can tune their instrument using the integrated tuner, they are assisted in the process of learning the most important chords using the lessons and test areas and can learn their favorite songs or can contribute with one in the community zone. So, the user has the opportunity to learn this beautiful instrument – the ukulele.

This application can be extended in each of the zones. In the future, it can have more types of tunings that can be helpful even for those who already have experience in playing the ukulele. The lesson area can be extended with more chords, and the test one can be developed for users to create their personalized test. The community can also be further developed in a social application so that any user can help directly any other user in their ukulele learning journey.

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Enhancing Summarizations and Presentations with Generative AI

Florin Olariu

Abstract

Generative AI is revolutionizing the development of summaries and presentations by providing exceptional speed, consistency, clarity, customization, and visual augmentation. It can quickly process large amounts of information, condensing them into summaries that save time and capture important details. Even for complex topics, these summaries are guaranteed to be understandable and unambiguous, thanks to advanced language models. AI tools can also modify the tone, style, and level of content information to suit different audiences' needs. Furthermore, generative AI helps create slides, graphs, and other visual aids that enhance and enrich presentations. Finally, AI upholds a high-quality standard by guaranteeing that the data supplied is accurate and devoid of human error. By employing ChatGPT and Gamma tools, together with Prompt Engineering techniques, we will focus on this research to study two essential aspects to improve the speed of task accomplishment: summarizing and presentations.

Keywords: Generative AI, ChatGPT, Gamma, Prompt Engineering.

1 Introduction

Since generative AI provides unmatched speed, consistency, and clarity, it revolutionizes how we approach summarizing and presentations. AI summarizes complicated subjects into understandable, succinct summaries, guaranteeing the accurate recording of even the most minor details. Suggestions produced by advanced language models are clear and concise, making them perfect for many uses. Whether used for technical reports or ordinary communication, AI-driven summaries save time and provide high clarity and coherence, automatically tailoring information to varied audience needs. Apart from written synopses, generative AI enriches presentations with customized image enhancement. Programs like ChatGPT [2] and Gamma [4] simplify the development of slides, graphs, and other visual aids, enhancing and increasing information engagement. Users can adjust the tone, style, and amount of content in presentations to make them more appropriate for different audiences and situations with the help of AI. Generative AI raises the bar for quality by ensuring correctness and reducing human error. This study will examine how using these tools in conjunction with prompt engineering can significantly increase the productivity and speed of work involving summarizing and presentations.

2 Case study: ChatGPT Summarization

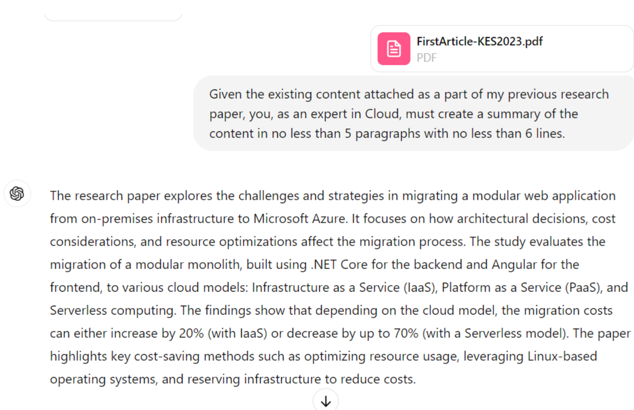

Figure 1. ChatGPT - Summarization prompt

Figure 1, based on this article [1], illustrates how, by using prompt engineering techniques, we can increase the pace at which we read competitive articles by simply asking the correct questions to elicit the appropriate response. The necessary components relate to constructing the proper prompt.

3 Case study: Create proper presentations by using Gamma tool

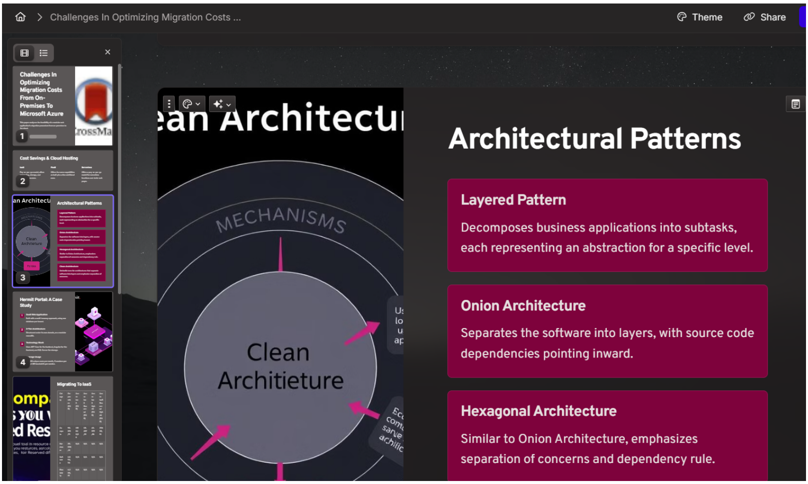

Figure 2. PPT presentation created with Gamma Tool

As shown in Figure 2, we can create PowerPoint presentations using the Gamma tool. It will be pretty simple for us to make presentations and modify the template. Our case studies illustrated each of these elements, which helps us increase our productivity.

4 Conclusion

With its speed, consistency, and clarity, generative AI is transforming presentations and summaries. It can condense complex topics into clear, short summaries that guarantee precise detail recording. Cutting-edge language models generate precise, succinct recommendations that save time and customize content to the needs of the audience. Apps like ChatGPT and Gamma simplify the development of visual aids and increase user engagement. This project investigates how generative AI might speed up and improve productivity in summarizing and presentations.

5 Future Work

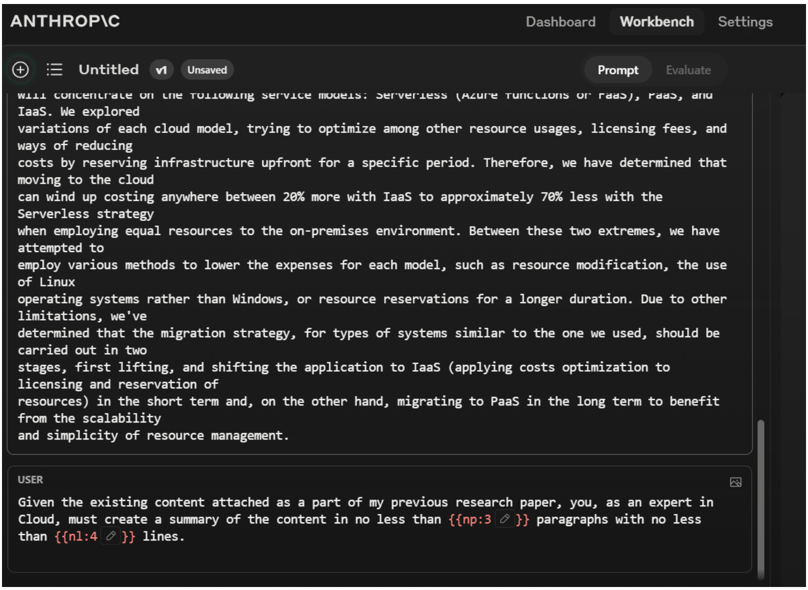

Figure 3. Anthropic and Claude tool

By contrasting Figures 1 and 3, we can see that Claude has several parameters we may set by providing additional information about the result, such as the number of lines and paragraphs. These features can also be defined in ChatGPT. The main difference is that Claude [3] allows you to supply these arguments as variables.

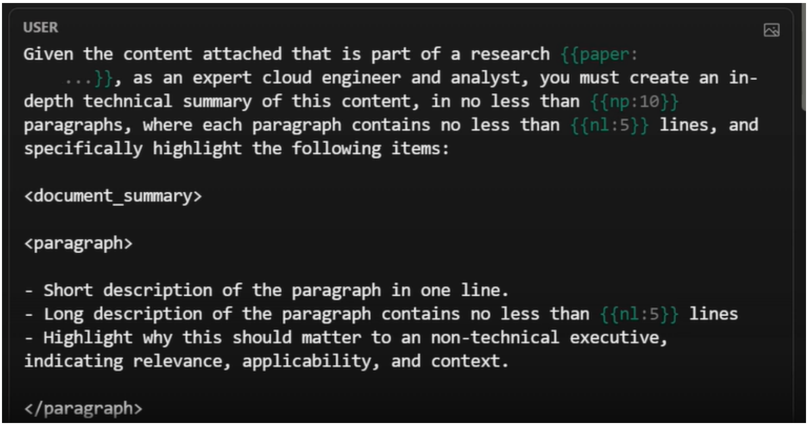

Figure 4. Multiple parameters in Claude

As we can depict in Figure 4, first, we have the document summary, which, as its name implies, instructs the engine to direct all output to this location, thereby guiding the GenAI model. Next, we specify that the output should have a one-line description and a lengthy description of at least five lines for each paragraph. This is a variable that we can modify at a later time. We also aim to emphasize the significance of the paragraph for a non-technical executive. We will examine Claude's options and compare ChatGPT's functionality based on these factors.

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Artificial Intelligence for Enhanced Medical Diagnosis: Developing Learning Models for Accurate Patient Diagnosis

Maria-Ecaterina Olariu

Abstract

This paper explores the integration of Artificial Intelligence (AI) in healthcare, focusing on its applications in medical imaging and diagnosis. It presents a case study evaluating generative AI tools in cardiovascular diagnostics, comparing their performance with expert analysis. The research highlights the potential of AI to augment human expertise in healthcare decision-making. Furthermore, it proposes future work on developing an AI-driven adaptive clinical decision support system based on European Guidelines. This system aims to personalize recommendations for individual practitioners while maintaining adherence to standardized best practices, potentially revolutionizing AI support in medical practice across Europe.

Keywords: Artificial Intelligence, Healthcare, Medical Diagnosis, Generative AI, Clinical Decision Support, European Guidelines

1 Introduction

Artificial intelligence is an umbrella term used for a number of technologies that are meant to mimic human understanding. Some of these tools are computer vision, machine learning, and natural language processing [1].

In recent years, there has been a growing trend to incorporate AI across diverse sectors, aiming to enhance efficiency and productivity in numerous processes. One field where AI has made significant strides is healthcare. The medical community has increasingly adopted AI-powered solutions to improve patient care and streamline operations. A prime example of this integration is the use of computer vision in medical imaging in areas such as X-ray Analysis, Oncology, and Cardiovascular Imaging. This technology helps doctors find problems that a person might miss, which improves the accuracy of diagnoses. Another relevant example would be integrating machine learning in diagnosis and treatment.

AI models can be trained on huge amounts of medical data, including health guidelines, research papers, and anonymized patient records. After training, one such model could analyze a new patient's medical records, including their symptoms, lab results, and medical history. Based on this analysis, the AI could suggest potential diagnoses or flag the patient for further testing, acting as a valuable support tool for healthcare professionals.

These applications demonstrate how AI is not replacing human expertise but augmenting it, allowing healthcare professionals to make more informed decisions and provide better patient care. As AI continues to evolve, its role in healthcare is likely to expand, promising a future of more efficient, accurate, and personalized medical services.

2 Previous Personal Work

Generative AI, or "gen AI," refers to deep learning models capable of producing complex material such as text, images, and audio [1]. While these models aid in various tasks including medical diagnosis, their increasing accessibility raises concerns about potential misuse in healthcare.

A case study [2] was done where we can see generative AI tools in such a context. This study aims to provide insight into their efficacy when confronted with actual patient cases involving prevalent cardiovascular conditions. This paper was done in collaboration with an experienced cardiologist, which also implied obtaining a range of authentic patient cases and medical data, ensuring the accuracy and reliability of our information. This partnership allowed us to thoroughly assess the diagnostic capabilities of the AI technologies and compare their results with expert medical analysis. We chose to employ images from Holter investigations due to their ability to present patient data concisely and precisely, facilitating the identification of potential sources of misunderstanding or inaccuracy in the AI models.

The study compared four AI systems: Gemini, Copilot, Claude, and GPT-4. All demonstrated good performance, with Gemini consistently outperforming others. Claude and GPT-4 excelled in acknowledging factors like medication non-adherence, while GPT-4 showed superior data analysis capabilities, particularly in processing multiple images simultaneously. This highlights AI's potential in longitudinal patient monitoring and complex case analysis.

3 Personalized AI in Healthcare

Personalized AI models can continuously learn and refine their responses by incorporating feedback from doctors, ensuring that the AI adapts to provide increasingly accurate and tailored recommendations over time.

There is significant ongoing research and development in the field of artificial intelligence (AI) models specifically designed to assist diagnosis, often leveraging European guidelines as a foundation. European guidelines, such as those published by the European Society of Cardiology (ESC), provide essential frameworks for developing and validating AI models. These guidelines often outline criteria for diagnosis, risk stratification, and treatment recommendations, which can be incorporated into AI algorithms to ensure their clinical relevance and effectiveness.

3.1 Future Work

It is the belief of this author that the future of this domain implies the development of an AI-driven adaptive clinical decision support system trained on European Guidelines that dynamically adjusts to individual medical practitioners' expertise and decision-making patterns while maintaining adherence to standardized best practices.

Improving the diagnostic accuracy is a core aspect of such a model. The system will use advanced AI techniques to analyze patient data, symptoms, and test results, providing practitioners with a ranked list of potential diagnoses and suggested next steps. The aforementioned techniques imply implementing a hybrid model combining deep learning for pattern recognition and reinforcement learning for adaptability to individual practitioners. The system will learn from each practitioner's decision-making patterns and adapt its recommendations accordingly, while still ensuring adherence to European Guidelines.

The innovation in this area comes from a variety of points, including the ability to adapt to each practitioner's expertise, specialization, and decision-making style. The system will update its knowledge base in real-time as new guidelines are published or updated, ensuring practitioners always have access to the latest evidence-based recommendations. Moreover, it will be able to work across multiple medical specialties, facilitating more comprehensive patient care and catching potential cross-specialty issues.

In regards to the research methodology, this author wishes to take into consideration 4 aspects: Data Collection, Model Development, Guideline Integration and Personalization. Anonymized patient data and practitioner decision-making patterns may be gathered from a collaboration with European hospitals and medical institutions. The model will be developed as a hybrid, combining convolutional neural networks for image analysis, natural language processing for text-based data, and reinforcement learning for adaptability. Integrating European Guidelines into the AI model's knowledge base will be done by developing an automated system to parse. The algorithm's design should be able to adapt the individual practitioner's past decisions and produce better recommendations in time. In order to prove the validity of the model, extensive studies will be conducted across multiple European healthcare systems to ensure the model's accuracy, adaptability, and adherence to guidelines.

4 Conclusion

This research has the potential to revolutionize how AI supports medical practitioners, offering a unique blend of standardized best practices and personalized assistance. It addresses key challenges in medical AI adoption and has the potential for significant real-world impact on patient care across Europe.

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A Comparative Analysis of Machine Learning Algorithms for Text Analysis

Alexandr Parahonco, Mircea Petic

Abstract

This article proposes a system of metrics for estimating text fetched from the Internet and selecting the one that should be further summarized. The research examines algorithms and software for determining user preferences, employing natural language processing (NLP) and supervised learning classification methods. An empirical assessment is conducted across academic, security, and non-security domains. The paper concludes with insights on the experimental results and the potential future of the implemented metric system.

Keywords: texts metrics, NLP, classification, supervised learning.

1 Introduction

Natural language processing and Machine learning algorithms are widely used in text analysis. These algorithms are able to analyze and interpret large amounts of text data, allowing for insights to be gained from unstructured information. Additionally, they can be used for tasks such as sentiment analysis, topic modeling, and text classification [1].

Learning is a fundamental human activity that entails the acquisition or change of behaviors, competences, and preferences. Diverse theories, including behaviorism and cognitive theory examine the mechanisms of individual learning. In contrast to humans, machines depend on data for their learning processes. Machine learning (ML), a subset of artificial intelligence, enables computers to autonomously learn and adapt. Machine learning emphasizes empowering computers to adjust their activities to enhance precision, commonly assessed by the rate of accurate results [2].

ML algorithms can be trained on large datasets to recognize patterns and make predictions without being explicitly programmed. This ability allows machines to continuously improve their performance and efficiency over time. It is about programming computers to optimize performance using example data or past experience. The goal is to automatically detect patterns in data and use them to make predictions. Machine learning is also applied to databases, statistics, engineering, and signal processing, among other fields [3].

In our research, we aim to develop metrics for decoding text meaning, extracting valuable information, and identifying the most relevant source to recommend to the user. Our findings are presented in Fig. 1.

Regarding the **structure**, our metric system is made up of four main parts:

- Quality and depth of content
- Context of metrics
- Relevance of the topic
- Authority

The components illustrated in Fig. 1 employ parameters to evaluate content based on its characteristics. Given the complexity of these features, each parameter may encompass multiple sub-parameters representing more straightforward attributes. For instance, text difficulty can be quantified through factors such as lexical choice, stylistic elements, syntactic intricacy, and textual coherence. Concurrently, readability can be gauged by considering the target audience's age demographic and overall comprehension ease. In this paper, we will implement the module "Context of the Sources".

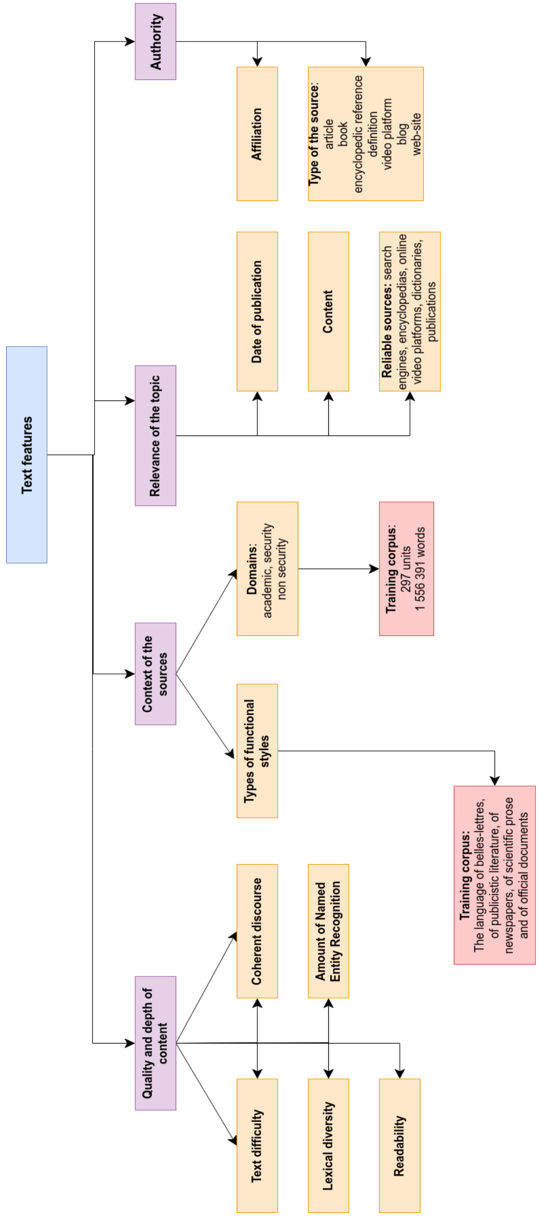

Figure 1. The metric system architecture

Our study focuses on three distinct domains: **academic, security, and non-security** (encompassing areas outside the security field). While our primary aim was to differentiate between academic and non-academic texts, we anticipate potential future expansion to accommodate various university faculties. To assess the adaptability of our system, we opted to incorporate an additional domain: security. This inclusion allows us to evaluate the system's capacity to handle diverse subject areas and potentially scale to meet evolving educational requirements.

2 Description of the experiment

Our research methodology involves primary, quantitative experimental analysis, structured in two distinct phases:

- Phase 1: text preprocessing, feature extraction procedure with the subsequent splitting the corpus into two parts: for training and evaluation.
- Phase 2: comparison of the classification models.

Based on our literature review, the most prevalent and effective supervised learning classification models are **K Nearest Neighbors, Decision Tree, and SGD Classifier**. In addition to these algorithms, our study examines the efficacy of **counting vectors** and the term **frequency-inverse document frequency** (TF-IDF) approach for feature extraction.

For our research, we have compiled a **corpus** comprising **297 documents** with a total word count of 1,556,391. The corpus is segmented into three domains:

1. Academic: 920,032 words
2. Security: 136,359 words
3. Non-security: 500,000 words

This dataset will be partitioned into two subsets: one for training our models and another for subsequent evaluation. This division allows us to assess the performance of our classification algorithms on previously unseen data, ensuring the robustness of our results.

3 Discussion and results

The results indicate that the TF-IDF feature extraction model outperforms Counting vectors. As shown in Table 1, 11 classification models achieved 100% prediction accuracy with TF-IDF, while only 7 models reached this level with Counting vectors.

Among the examined models (K-Nearest Neighbors, Decision Tree, and SGD Classifier), the SGD Classifier with TF-IDF feature extraction consistently demonstrated the highest accuracy. Counting vectors, on the other hand, proved to be less effective and reliable.

Table 1. The comparison of the feature extraction models

$Model$	$Accuracy 100\%$	$Accuracy 100-95\%$
$TF-IDF$	11	19
$Countingvectors$	7	17

The results obtained from our experiments validate the quality and suitability of the developed corpus for implementing our metric system for text classification. Specifically, the corpus effectively distinguishes between academic, security, and non-security domains. To further refine our findings, we recommend conducting a final validation using manually selected texts from diverse sources. This step will provide a more precise assessment of our results.

4 Conclusion

In conclusion, we have successfully developed a metric system and conducted initial testing of its "Context of the Sources" module. Our experiment involved two stages: text preprocessing and feature extraction, followed by corpus splitting for training and evaluation. By comparing various classification models, we determined that the SGD classifier using TF-IDF features is the most effective for domain identification.

Acknowledgments. This article was written within the framework of the research project 011301 "Information systems based on Artificial Intelligence".

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A Simple CAD System for Faculty Timetabling Design

Andrei Rusu, Elena Rusu

Abstract

A computer-aided design (CAD) system is proposed in order to help in designing the timetabling of courses in a faculty. It is based on already used technologies such as Google Spreadsheets and Microsoft Excel and tools built-in in them: Server JavaScript, Visual Basic for Applications (VBA). The idea is to use a single sheet "Collision" in a spreadsheet in Microsoft Excel to deal with all information related to the timetable, and then, by parsing its cells, to generate all needed timetables for students, teachers, rooms, etc.

Keywords: Timetabling, CAD systems, VBA, JavaScript, mathematical models.

1 Introduction

The development of a timetable for courses in a faculty is a difficult task that in our days, in many faculties in Moldova, are still done in an old manner using sheets of paper for groups of students, for professors, and for rooms. Any single record related to the timetable of a group of students, or of a professor, or a room requires to make the corresponding modifications in all related timetables for students, for professors, and for rooms.

There are many available software systems on the market which can help to automatize the process to obtain quickly a qualitative timetable for the courses in a faculty, but for different reasons they are not used.

Among reasons we meet the following: the systems are difficult to use, they are costly, they do not complain with the requirements in the faculty, the result is not one that is expected, and so on. The surveys for timetable for universities are available in [1, 2].

The goal of the present paper is to present a relatively simple CAD system based on tools already known by the majority of faculty members, such as Microsoft Excel and Google Spreadsheets together with their built-in tools, such as VBA (Excel) and JavaScript (Google Docs).

2 The problem

The timetabling problem is a problem related to to the optimal allocation of resources and it has different variations. We consider the following one (a usual case in a faculty in Moldova or Romania). We consider for simplicity that all course activities are organized in the same way during a week.

There are some rooms $S = \{S_1, \ldots, S_{N_S}\}$, each one with some properties, such as number of seats, available projector, number of available computers, etc.

There are also some study groups of students $G = \{G_1, \ldots, G_{N_G}\}$.

There are some professors $P = \{P_1, \ldots, P_{N_P}\}$.

Course activities take place in any week-day $Z = \{Lu, Ma, Mi, Jo, Vi, Sa, Du\}$, and in some standard time interval $T = \{8-10, 10-12, 12-14, 14-16, 16-18, 18-20\}$. By a timeslot we understand the tuple of a given day and a given time interval.

There are also a list of courses $D = \{D_1, \ldots, D_{N_D}\}$ and type of activity associated to such courses $\{Course, Seminar, Laboratory\}$ that are specific (are allocated) to each group of students.

Each professor is allocated to teach some courses of the corresponding type to some groups of students. Each such tuple $\{(P_i, G_j, D_k)\}$ is called a timetable event or teaching activity.

The problem is to allocate day, time interval, and room to each timetable event such that it is complained with the so-called hard restrictions: (a) each professor is involved at most in one event for a given timeslot, (b) each group of students is involved at most in one event for a given timeslot, (c) at most one event happens in a room at a given timeslot.

There are also so-called soft restrictions which are related to the preferences of students, professors, and administration. Often some soft restrictions are actually hard ones (preferences of administrations, availability of some professors, etc).

The problem is to find out a solution that respects all hard restrictions and as many as possible soft restrictions.

3 Proposed solution

The proposed solution is to use google spreadsheets, and register all events in a sheet "Collision", each column of which is the room, and each row is the timeslot (see Fig. 1).

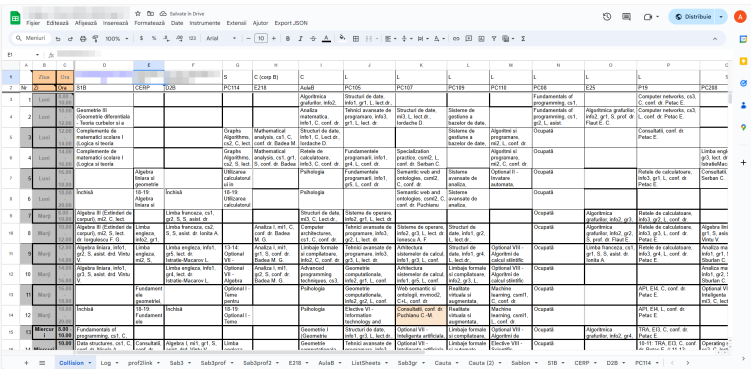

Figure 1. Collision sheet

Then use a Google script that will verify compliance with all your restrictions every time you register an event in Collision. Thus you finally get a feasible timetable. Then download the spreadsheet and use VBA to parse the Collision and generate all necessary timetables for students, for professors, by rooms, etc.

4 Conclusion

The proposed system is relatively simple to implement in a faculty, but it is not suitable to be used at a university level, or for other timetabling systems. The system is designed to help with the construction of the timetable, but it can be also adapted to automate some more things based on methods described in [2].

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Integrated Decision Intelligence Approach to Catalyse the Valorisation of Knowledge Assets

Iulian Secrieru, Elena Gutuleac, Olga Popcova

Abstract

IDI is a new direction aimed at improving decision-making efficiency by using advanced methods and technologies (including artificial intelligence, machine learning and data analysis). Developing digital support for IDI would bridge the gap between traditional methodologies based on knowledge, skills and expert experience, and the capabilities offered by data-driven analysis. In this paper, we present how the IDI approach can be used for previously created knowledge assets in the medical diagnostics domain to catalyse their valorisation as a source of reasoning.

Keywords: decision-making process, decision intelligence, knowledge-based decision, data-driven decision, integrated approach

1 Introduction

Integrated Decision Intelligence (IDI) is an approach to the decision-making process that leverages the collective power of human knowledge and empirical data to boost innovation, optimize processes, and navigate the complexity of a rapidly evolving landscape. The general scope of the proposed concept is the development of methods and algorithms for the acquisition, storage, pre-processing and inference of professional knowledge and data from various domains, as well as visualization and explanation of the generated conclusions. All these represent components of the knowledge engineering methodology aimed at decision support and adjusted to the new approach based on both professional knowledge and skills, as well as data/precedents, describing the perspectives of the problem domain.

IDI methods allow identifying regularities, trends and interconnections in complex data sets that may not be evident through traditional analytical approaches. IDI facilitates the integration of expertise and data from different scientific disciplines. This interdisciplinary combination gives us the possibility to approach problems comprehensively, facilitating the emergence of new ideas and discoveries at the intersection of different domains.

Given the explosive growth of IDI technologies and the sheer volume of accumulated data, it is expected that the demand for IDI applications will grow rapidly.

IDI has significant commercial potential in a variety of fields, including public health, medical diagnostics, and treatment planning.

2 IDI Benefits and Risks

Integration of statistical methods, computational models, and knowledge specific to a concrete scientific domain improves the analysis of complex data, enabling more accurate predictions and correct conclusions. Continuous feedback loops in IDI systems assist researchers in learning from the results of decision-making, improving models and strategies. Also, IDI helps to optimize the allocation of resources by identifying areas that need more attention.

Medical diagnostics was selected as the primary area of development of the proposed approach.

Conducting a comprehensive review of the existing literature [1-3] showed that IDI offers many benefits, but there also exist risks associated with its implementation. Let us mention the most important of them and possible mitigation ways.

1) Inaccurate or unreliable data can lead to inaccurate results during the decision-making process. The solution here would be to establish standards for data management, authentication, data curation procedures and regular audits.

2) Lack of standardization of data formats and decision-making models may hinder their interoperability. To mitigate this risk, the data integration platforms should follow the recognized standards.

3) Unauthorized access or data leakage may compromise confidential information. Robust security protocols should be implemented, including encryption, access control, and regular security audits.

4) The excessive complexity of a system can lead to confusion (misunderstanding) or dependency on technologies without critical evaluation of them. The solution here would be a user-friendly interface and a clear presentation of the results.

3 Revision of knowledge assets based on IDI

As part of the re-inventory of professional knowledge resources, data sources, and available software relevant to the selected domain, a comprehensive revision of existing knowledge assets is carried out. There were analyzed expert knowledge bases (extensive ones) in the field of ultrasonographic diagnostics of the hepato-pancreato-biliary area [4] (129 nodes for liver, 149 – for pancreas, 115 – for kidney, 114 – for spleen); the core of knowledge bases restricted by some principles: limited by the examination time, examined area, purpose, and conditions of investigation, etc. (30 nodes for hemorrhagic shock, 63 – for EFAST); scoring systems developed for differentiated diagnosis of liver cirrhosis [5]; triage based on vital signs for patients in multiple casualty disasters [6] (7 vital signs as basic characteristics, 4 triage categories).

Subsequently, knowledge (facts and decision rules) were identified. They formed a set of knowledge gaps that can be addressed by analyzing by data collections, which are dependent on real-time changes (short or medium) and which would be good to manage both by classical expert knowledge engineering methods and by data-driven approaches (based on data analysis).

Working sessions with experts in the field of medical diagnostics were organized in order to evaluate the perspective of the IDI approach and the accessibility of data and knowledge sources, possible problems and risks were discussed.

As a result, several sub-domains with good potential for research and implementation of IDI solutions were determined. Subsequently, they were ordered by potential and possible risks (high potential and minimal risk): 1) differentiated diagnosis of liver diseases; 2) scoring systems used under specific conditions (time constraints, incomplete data, limited technical connections, etc.); 3) new decentralized technologies aiming at attracting new independent sources of diagnostic data. For the first two, there were created curated data sets – ready for analysis. Later, validation sessions are planned to be organized.

4 Conclusion

After study of both the literature and the opinion of experts in the domain of medical diagnostics, a drastic decrease in the development of new scoring systems, as a tool for assessment and evaluation, was revealed. Even worse is that the use of old scores in practice has also decreased [7]. The main reason is that the new ones are based either on professional expert knowledge or data analysis, while the old ones are based on expert or theoretical knowledge and do not have continuous support in access to real up-to-date data. On the other hand, the detected problem from the above-formulated description is a good test for the IDI approach as a proposed solution.

Acknowledgments. The institutional project 011301 'Information systems based on Artificial Intelligence' and the project in the framework of stimulating excellence in research 24.80012.5007.24SE 'BOOSTing decision making: applying an integrated Decision Intelligence approach to overcome the limitations and challenges of traditional scoring systems' have supported part of the research for this paper.

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Early Detection of Cyber Attacks Using Decoy Systems – Analysis of Deception-Based Cyber Security Strategies

Mădălina Spiridon, Lenuța Alboaie, Dragoș Gavriluț

Abstract

Deception technology is a modern security solution that provides specific actions against malicious activities and is a relevant component for a complete approach to cyber security. The methodology behind this solution involves the creation and deployment of decoy assets, easy to configure and highly attractive, encouraging attackers to engage with them. The dynamic aspect of continuously producing and updating these decoy assets has been a major discovery point in this field.

Keywords: deception, security, innovation, engage, dynamic.

1 Introduction

Ransomware and info stealers continue to affect a growing number of victims, demanding significant ransom payments to restore compromised data and pledging not to release sensitive data on the dark web [1]. Although these threats are not a new concern, attackers have adapted their techniques to circumvent current security solutions [2]. In response to these persistent malware threats, it has become essential to develop a proactive and efficient strategy capable of detecting and responding to attacks almost in real-time. The scope of deception technology is the protection against ransomware attacks and info stealers, being an useful technique to have in place once a suspected event has happened. Additionally, these technologies slow down an attack by increasing the computation time for file encryption.

2 Deceptive Technologies

Deceptive technologies, such as those patented for detecting unauthorized access, often involve traps that mimic real network resources and deploy decoy drivers to attract attackers [3]. Notable examples include FortiDeceptor's AI-driven honeypots for isolating attacks [4], Fidelis's deception mechanisms to expose attacker tactics [5], and Acalvio's ShadowPlex system, which deploys various decoys across digital assets [6].

Decoy technologies serve as an advanced defense layer against cyber threats like ransomware and data theft. Decoy system use deceptive artifacts, which are fake assets mimicking real ones, to lure attackers, and any interaction with these monitored decoys signals potential malicious activity. The decoy module integrates with broader security ecosystems, enhancing protection by working alongside existing security solutions such as SIEM[1] and endpoint protection platforms.

3 New Insights in Decoy Systems

In this paper, we are proposing a module that incorporates methods to collect indicators of compromise, ensuring that when malicious activity is detected, the content of the compromised file is preserved. This allows it to be safely detonated later in a completely isolated environment, where it can be analyzed in depth to discover the adversary's tactics and to better understand the attack kill-chain. To highlight the capabilities of the decoy module, a series of tests were performed using ransomware and info stealer samples with different scenarios such as: commands and key-exchanges from a C2 server, code injection mechanisms, which use memory stores or which involved integration with

[1] Security Information and Event Management

Telegram bots for publishing and selling stolen data. Additionally, tests were conducted using ransomware specialized in identity theft aimed at stealing banking information and browser data. During tests involving

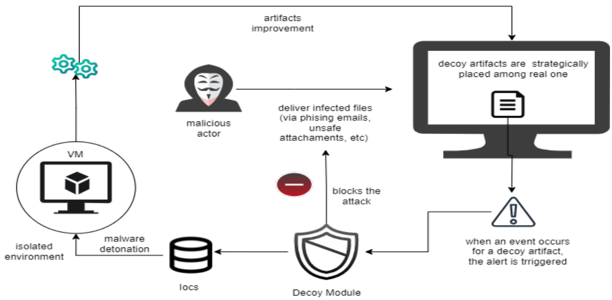

Figure 1. Architecture

ransomware and info stealer samples, the module demonstrated its effectiveness both in terms of detection speed, with an average detection time (the duration it takes for the system to identify a potential threat or suspicious activity) of 2-3 seconds, and the time it took to block the attack, ensuring that no real files were affected, only decoy artifacts. During the tests, no false positives have been observed.

	non decoy files	**encr non decoy files**	**decoy files**	**encr decoy files**
#1	13	0	9	2
#2	13	3	18	9

Table 1. Testing Results

The decoy module aligns with market protection standards and adheres to the techniques and tactics prescribed by the MITRE [7] organization while also incorporating innovative components.

4 Conclusions

In this period when attackers are gaining more and more ground and are developing using advanced techniques, the implementation and use of an integrated solution with deceptive technologies is an additional protection measure. The decoy technology will have a considerable impact in improving defense and is expected to gain popularity in the market due to its beneficial contributions in terms of early detection of cybercriminals. The role it plays in improving security can be major because it can easily detect the newest types of ransomware and maintain protection against sophisticated attacks.

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Orion – Smart Generation of Work Schedules

Tudor Tescu, Adrian Iftene

Abstract

In today's work environments, many businesses, especially those active in the HoReCa (Hotel, Restaurant, and Café/Catering), stay away from fixed schedules. Since most of such companies are open every day, there exists the problem of employees having to work weekends and, by having a variable schedule, no employee will have to work every weekend. The use of variable schedules, however, creates the burden of creating an optimal arrangement of shift assignments very often, and business owners are tasked with spending many hours considering each employee's preferences, capabilities, and contractual details. The present paper presents a solution that comes to the aid of these owners and helps them manage their employees' activities more efficiently.

1 Introduction

While there already exist a vast amount of applications dedicated to helping business owners and their teams create schedules faster and easier, most of them only allow the creation of schedules and the assignment of employees, with no extra details or capabilities. The applications that can generate schedules often generate an arrangement of shifts and assignments that approximately match previously created schedules but without any customization. Thus, the idea of creating a better application with powerful features for business owners emerged.

The selling point of this application is the automatic schedule generation feature. This functionality works hand-in-hand with the employee profile customization capability, allowing the employer to set specific conditions for each employee, such as limiting them to certain shift areas (for example, a cook should only work in the kitchen and should not be assigned to the serving area), set the maximum number of hours allowed per day, give the employee several days off after working for several consecutive days (many businesses such as restaurants prefer to have their serving staff work for two consecutive days, often for 12 hours each, and then give them one or two days off), or to explicitly provide the employees time off. As for the employees, they can select (either manually or from the defined shifts in the latest and upcoming schedules) their unavailability intervals. These features work hand-in-hand to offer the employer and their team an efficient method of generating schedules based on imposed conditions.

2 Similar applications

Deputy[1] is an app that offers scheduling features and the ability to create and swap shifts. Furthermore, it tracks attendance and working hours and helps businesses comply with state laws. Deputy also offers an automatic generation feature, which, however, only creates an approximate arrangement of shifts based on previously created schedules.

Coast[2] is an app that markets itself as an alternative to Deputy. It offers the same features as the aforementioned Deputy app while also giving employers the means of messaging their team.

Like the previously mentioned applications, **Homebase**[3] aims to help business owners with scheduling. It offers advantageous features such as calendars, time off, and compliance with local law, but it lacks automation, which is the key point of the application presented in this paper.

[1] www.deputy.com
[2] www.coastapp.com
[3] www.joinhomebase.com/employee-scheduling/

3 Architecture and Implementation

Orion, the application presented in this paper, consists of the frontend application and the backend services. The frontend application is built using Angular and Taiga, and the backend consists of two separate microservices - Identity, which is responsible for handling authentication, and Administrative, which is responsible for handling everything related to the administration of the business and the schedules. Both of the aforementioned microservices are built following the Domain Driven Design architecture [1] and the Test Driven Design pattern to ensure consistency and reliability. Furthermore, to provide increased security, performance, and scalability, the backend services follow the CQRS design pattern. The infrastructure layer holds and implements abstractions of services declared in the application layer. Furthermore, it holds the implementation of the repository and manages the connection to the database. While the layer itself does not perform many actions, it is crucial for implementing a backend application following the DDD approach, as it is also responsible for most configurations that involve the use of services or the handling of credentials. The presentation layer is the layer responsible for communication with the outside world. In this case, the presentation layer is the intermediary between the Angular frontend application and the rest of the backend layers. For the development purpose of Orion, the presentation layer consists of a series of Azure Functions HTTP Triggers, which are methods that intercept and handle HTTP requests by sending commands or queries to the respective handlers. While Azure Functions also supports other useful features, such as a Service Bus (which would have been helpful in this application in specific scenarios), only the HTTP triggers have been used, as the other features would have required more infrastructure and costs.

3.1 Use Cases and Benefits

While the application's selling point is the intelligent generation feature, it was designed to be user-friendly and easily understandable.

Thus, the most common use case is viewing the created schedules in a human-friendly calendar format. Furthermore, statistics and information for each employee can be observed, such as contract hours, worked hours, and various conditions. The feature can be observed in Figure 1 below.

Figure 1. The schedule overview feature

The chosen design displays all the necessary information regarding the schedule. Assigned shifts can be easily observed and are color-coded uniquely to each employee. Furthermore, unassigned shifts have a distinctive look, making them difficult to miss. Statistics can be seen for each employee, making it easier to identify potential issues.

4 Conclusion

The presented application can serve as a powerful tool in a business owner's arsenal by allowing them to easily schedule their employees, either manually or by leveraging the built-in automation feature. During the development of the presented application, we faced various challenges, ranging from choosing the tech stack, architecture, and libraries to figuring out the best possible way of generating shift assignments.

While this application can be considered complete, well-structured, and reliable, further improvements can be made in the future. First off, additional employee conditions can be created, such as shift pairing restrictions to promote better dynamics between certain employees, mandatory days off, and various others. Additionally, certain pages could benefit from redesigns, such as the page that displays a specific schedule. This particular page has proven to be a great challenge, as developing a calendar from scratch would have been a challenging and highly time-consuming task, and libraries that could have provided a better experience are, unfortunately, hidden behind paywalls, which would have been detrimental. Last but not least, the generation algorithm would also benefit from further research and improvement, as an algorithm can never be too efficient. By creating further versions of the algorithm, more cases could be considered, such as the additional conditions mentioned earlier.

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Exploring Interactive 3D Object Interaction by Integrating Augmented Reality and Voice Control

Inga Titchiev, Olesea Caftanatov, Daniela Caganovschi, Dan Talambuta

Abstract

This paper explores the integration of augmented reality (AR) and voice command technology in educational settings, specifically focusing on the interactive control of 3D object models. By combining AR with voice commands, students can engage in immersive learning experiences, allowing them to visualize and manipulate digital objects in real-time. This approach enhances understanding of complex concepts, increases student engagement, and offers accessible learning opportunities for all students.

Keywords: augmented reality, voice command, 3D interaction

1 Introduction. Background research

Over the past three years, our team has conducted extensive research into the integration of Augmented Reality (AR) technologies within the educational field. This exploration has involved the design and implementation of various learning style scenarios aimed at enhancing user engagement with educational content [1, 2]. Through this process, we have gained valuable insights into both the potential and the challenges associated with AR applications in education [3]. Our primary objective has been to develop AR applications that cater to diverse learning styles, thereby increasing user engagement. By creating immersive and interactive learning environments, we sought to make educational content more accessible and appealing to students. This approach has led to the development of scenarios that are not only visually engaging but also pedagogically effective, leveraging the unique capabilities of AR to provide enriched learning experiences.

We developed mobile applications for learning both math and languages. For language, learning specifically includes apps like the Etymology app (see Fig.1), which uses augmented flashcards,

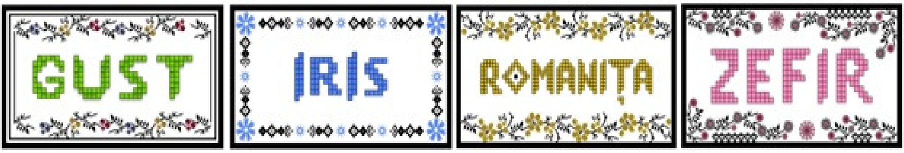

Figure 1. Personalized cards for learning etymology and morphological terms

and the Multiword Expressions app that, by scanning clock card, generates a list of MWEs related to such terms as "clock", "watch", "time", and "weather" (see Fig. 2).

Figure 2. Multiword Expressions app showing the example "vremea patului" with its English translation, "It's Bedtime"

Throughout this journey, we have encountered a range of challenges, which can be broadly categorized into four areas: technical, usability, ethical, and practical considerations of AR applications.

Our applications have been tested on a sample of over 200 pupils and students. Feedback from these users has been instrumental in refining our applications, leading to several upgrades that have improved both the functionality and the user experience. This iterative process of testing and enhancement has ensured that our AR solutions remain responsive to the needs and preferences of students. According to the responses to the 10th quiz question about which scenarios students enjoyed the most, 74.1% of students from Moldova State University (MSU) preferred visualizing 3D objects, and 55.6% enjoyed interacting with animals. See the diagram in Figure 3 for more details.

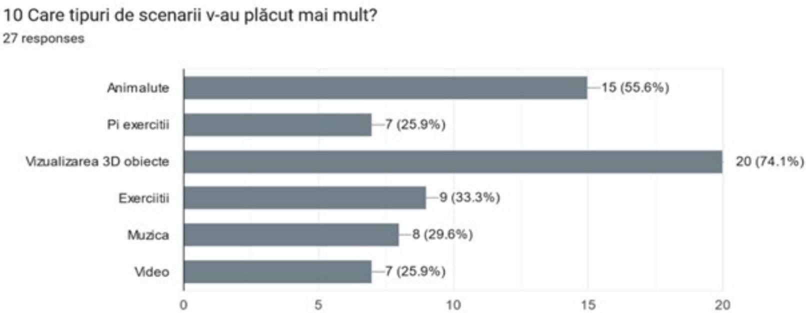

Figure 3. Diagram illustrating responses to the 10th quiz question

Both animal interaction and 3D object manipulation garnered significant interest, not only from MSU students but also from students at Alecu Russo University. When we initially presented the augmented scenario Wolf (v.1.0) to students at Alecu Russo University, they showed interest but found it relatively simple. Based on their feedback, we enhanced the scenario's complexity by adding interactive features to the 3D model. We introduced four buttons that allow the wolf to perform actions such as running, howling, lying down, and fighting, each accompanied by sound. These updates are illustrated in

Figure 4 (Wolf v2.0).

Figure 4. Updated version of the Wolf scenario

Building on our previous work, this research seeks to explore a new recommendation provided by 6th-grade pupils. These students expressed a desire to interact with 3D animals via voice command, suggesting a new avenue for enhancing the interactivity and engagement of AR applications. This research will analyze the feasibility and educational impact of incorporating voice-controlled interactions within AR environments, with the goal of further personalizing and enriching the learning experience for young learners.

By integrating voice command functionality, we aim to create a more dynamic and responsive AR experience that aligns with the evolving expectations of tech-savvy students. This development represents the next step in our ongoing efforts to harness the full potential of AR in education, making learning not only more effective but also more engaging and enjoyable.

2 Voice recognition technology

Voice recognition, also known as speech recognition [4], is a technology that enables a computer or device to identify and process spoken language. It allows users to interact with technology through voice commands, converting spoken words into text or executing specific actions based on the commands. Speech recognition technology has become a central element in millions of homes worldwide, enabling devices to execute commands based on voice input. However, its applications extend far beyond simple voice commands, reaching into various sectors where it makes a significant impact.

The integration of voice recognition in augmented reality is poised to transform the way we interact with digital content and services. By leveraging the power of hands-free interaction, voice-enabled AR is opening up new possibilities for productivity, accessibility, and innovative user experiences. In AR applications, voice recognition allows users to control digital elements without relying solely on physical controllers or touch-based interfaces. This hands-free interaction enables users to seamlessly navigate virtual environments, manipulate 3D objects, and perform various tasks using only their voice, a very useful thing both for people with a preferential auditory learning style and for people with physical disabilities.

By simply speaking commands, students can rotate, scale, or animate 3D animals, exploring different perspectives and behaviors. For example, a student might say, "Make the wolf roar," and the AR system would animate the 3D wolf accordingly, providing an interactive and auditory learning experience. Imagine a biology class where students are learning about animal behavior. With the combination of AR and voice commands, students could bring a virtual safari to life in their classroom. They could command a 3D wolf to walk, observe how a sparrow flies, or explore the movements of a fish in water. This interactive approach allows students to explore the characteristics and habitats of various animals in a controlled environment, fostering a deeper understanding of the subject matter.

3 Integrating Voice Command tech in AR app

Voice command technology further enhances the AR experience by allowing students to control and interact with 3D models using their voice. This hands-free approach not only makes the learning process more engaging but also more accessible, particularly for students with physical disabilities.

In the wolf scenario, voice commands such as ”howl”, ”run”, ”attack”, ”stop”, ”sleep” were integrated, see Figure 5.

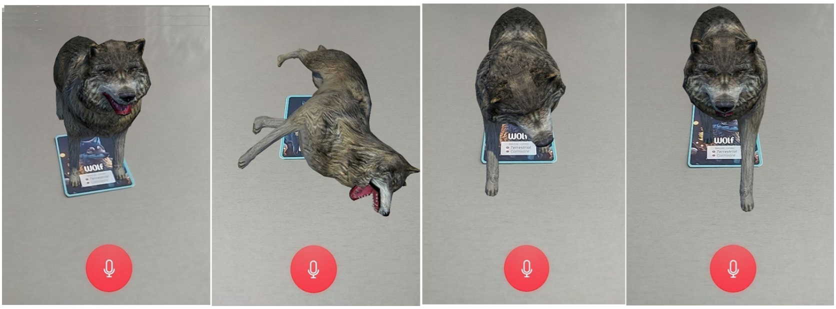

Figure 5. Updated with voice command version of the Wolf scenario

Also in the bear scenario, voice commands such as ”sleep”, ”run forward”, ”attack”, ”eat” were integrated, see Figure 6.

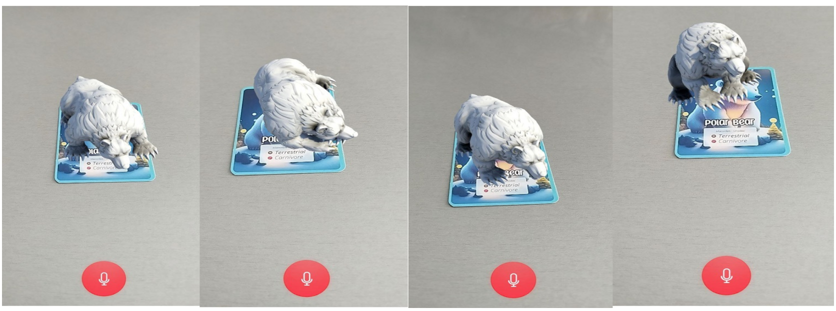

Figure 6. Updated with voice command version of the bear scenario

While the potential of AR and voice commands in education is immense, there are also challenges to consider. Implementing these technologies requires significant investment in hardware and software, as well as training for educators to effectively use these tools in the classroom. Additionally, there may be concerns about screen time and the impact of prolonged exposure to digital devices on students' health. However, with careful planning and thoughtful integration, these challenges can be managed to maximize the benefits of this innovative approach to education.

4 Conclusion

The combination of augmented reality and voice command technology represents a powerful tool for modern education. By enabling students to control 3D objects through simple voice commands, educators can create an immersive and interactive learning environment that enhances understanding, engagement, and accessibility. As these technologies continue to evolve, they hold great promise for transforming education and preparing students for a future where digital and physical worlds are increasingly intertwined.

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Implementing Convolutional Neural Networks and Vision Transformers for Satellite Image Processing

Matei-Octavian Țurcan, Olesea Caftanatov

Abstract

This paper explores the application of Vision Transformers (ViT) for satellite imagery classification, inspired by the widespread use of transformers in language processing and related computer vision research. The study involved training a ViT model on the EuroSAT dataset and comparing its performance with ResNet50. Additionally, a hybrid model architecture integrating CNNs with ViTs was proposed to leverage the strengths of both approaches.

Keywords: Visual transformers, convolutional neural networks, satellite image, datasets, image recognition.

1 Introduction

In recent years, deep learning has revolutionized the field of computer vision, with convolutional neural networks (CNNs) playing a pivotal role in this success [7]. CNNs have excelled in visual tasks largely due to the powerful inductive bias of spatial equivariance encoded by their convolutional layers. This capability has enabled CNNs to learn general-purpose visual representations that can be easily transferred across various tasks, resulting in consistently strong performance.

However, a significant breakthrough in the field has emerged with the introduction of Vision Transformers (ViTs) [1]. These models, which are adapted from Transformer neural networks originally designed for language processing, have demonstrated that they can achieve performance on par with, or even surpass, CNNs in large-scale image classification tasks. Unlike CNNs, which aggregate spatial information through convolution, ViTs use self-attention mechanisms to capture information across the entire image, operating in a manner more akin to how Transformers process language data.

Despite their success on large-scale datasets, ViTs often struggle with performance when trained on smaller datasets. In contrast, CNNs like ResNet-18 can achieve high accuracy on small datasets and generally improve as the number of parameters and layers increases.

To address these challenges, we propose a hybrid architecture that combines the strengths of both Convolutional Neural Networks (CNNs) and ViTs. CNNs are well-suited for extracting local features from small and imbalanced datasets, while ViTs excel at capturing global context and complex patterns. By integrating these two approaches, we can achieve more robust and accurate classification across a broader range of satellite image datasets, effectively leveraging the strengths of both models to overcome the limitations inherent in small satellite datasets.

2 The EuroSAT dataset

The dataset used in this experiment is EuroSAT, containing 10 image classes provided by the Sentinel-2 satellite mission. The set is freely available for use under an open licence, as are the images supplied by the satellite. The Sentinel-2 satellite system provides approximately 1.6 terabytes of compressed imagery daily. The data is collected from several countries across Europe to different degrees, dominated by western European countries. The classes are as follows: Forest, River, Highway, Annual Crop, Seas or Lakes, Herbaceous Vegetation, Industrial, Residential, Permanent Crop, and Pasture. This breadth of image classes provides ample opportunity to create specific applications for land cover and use, particularly in agricultural or wild environments with seven out of three classes being reserved for such land uses.

This set contains a relatively large number of images inside each

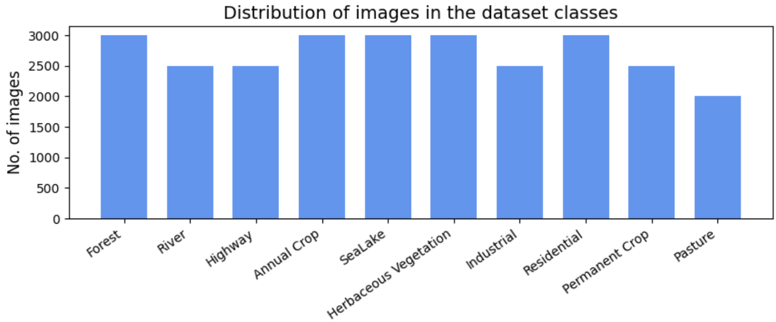

Figure 1. Number of images per each class inside the dataset

class, ranging from 2000 to 3000 as seen in Figure 1, containing a total of 27000 images with a 64 x 64 pixel size. The images were selected based on their clear visibility at a resolution of 10 meters per pixel and were sufficiently covered by the European Urban Atlas. This data set is available in a multi-spectral version, containing 13 spectral channels, and a regular 3 channel RGB set. While the additional spectral channels can be helpful to combat atmospheric effects and provide additional information undetectable by humans to a classification model, this greatly increases the input shape of the images, and consequently, increasing the total training time. As such, the 3 channel set was used in this paper.

The EuroSAT dataset is valuable because it includes various types of land cover, which are essential for urban planning, environmental monitoring, and resource management. By examining changes in urban areas over time, the dataset aids in tracking urban sprawl, identifying urbanization trends, and promoting sustainable city planning. Additionally, the dataset helps monitor crop health, estimate yields, and manage agricultural resources more effectively by analyzing different land cover types and their temporal changes.

3 Vision Transformers

Vision transformers (ViT) are somewhat infrequently used in the field of computer vision for classification tasks, but provide an interesting proposal in terms of handling large datasets with ease. Furthermore, the self-attention trait can prove to be quite useful for global feature detection, as seen in numerous natural language processing applications. The visual equivalent of tokens in the vision transformer architecture is echoed by patches, formed by splitting the desired image into many subsections.

3.1 The Vision Transformer Architecture

The way vision transformers are designed closely resembles how natural language processing (NLP) transformers are created, but slightly modified to accommodate the usage of images in its input.

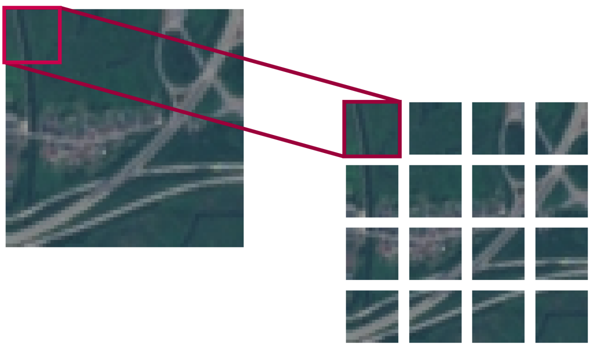

Figure 2. Creation of patches from an input image

Following the architecture of Dosovitskiy et al. [1], the input image is patched into subsections, as seen in Figure 2, which are the visual equivalent of tokens in NLP transformers. The patch size is also defined as a hyper parameter able to be changed in order to determine how broad or narrow the focus should be on the model's local attention.

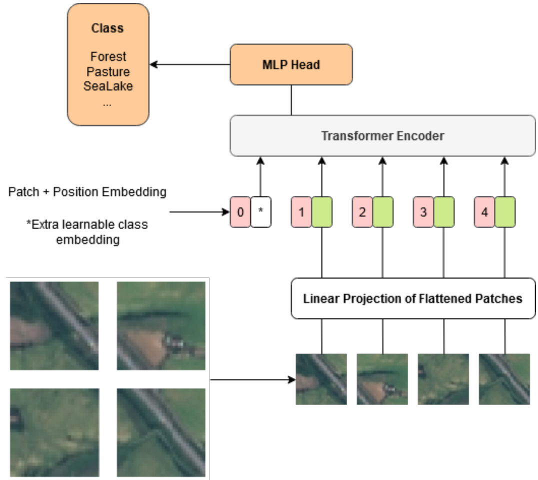

Figure 3. The vision transformer structure

The patches are flattened and are sent into the encoder along a position embedding to determine the patch's position in the original image, and the class embedding used for classification later on in the Mulit-Layer Perceptron (MLP) head.

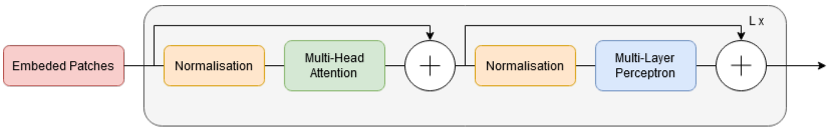

Figure 4. The transformer encoder architecture

Inside the Multi-Head Self-Attention block is the mechanism by which the model is able to focus on the important details found in the image, enabled by learning the relationships between the patches which are fed into the encoder.

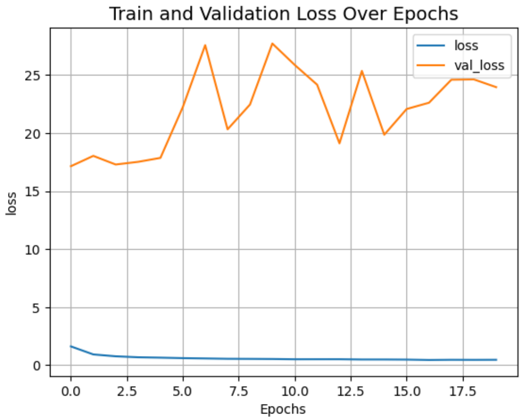

Figure 5. Vision transformer validation loss

3.2 Using ViTs for direct classification

Dosovitskiy et al. [1] have used ImageNet-21k and the JFT-300M datasets to pre-train the ViT model stating a great degree of success. However, both ImageNet-21k and JFT-300M are very large datasets with over 21000 and 375 million general classes respectively compared to EuroSAT. Additionally, both sets have a total number of pictures several orders of magnitude higher than the EuroSAT dataset, with the images consisting of a larger resolution too. All these characteristics result in a much more lessened performance proposal in computer vision for ViTs in the context of satellite imagery.

4 Testing ViTs on EuroSAT

The previously mentioned lack of performance due to the small scale of the dataset, is reflected by its performance across multiple test runs of the model. Initially, the model was very unstable and erratic in output; however, slightly changing the model has resulted in a low 11% validation accuracy.

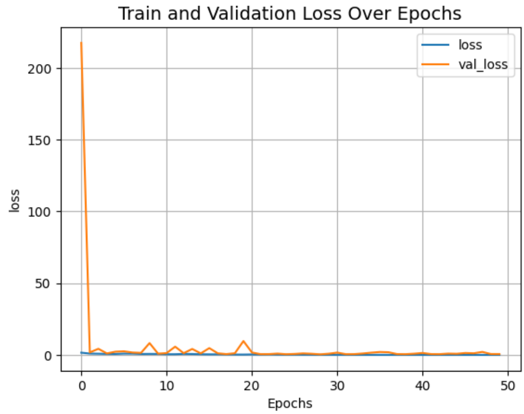

Figure 6. ResNet50 validation loss

The graph in Figure 5 indicates that the ViT has issues with a lot of overfitting, and this is further proof of the data-hungry characteristics of the model along the other factors discussed by Zhu et al. [6].

Training the model from scratch to only classify image data proves to be complicated and unreliable in performance, but this could be mitigated by creating an even larger dataset. This shows that there is a compromise to be had regarding the retrieval of satellite data and ViT performance, as attaining such detail from satellite imagery can become cost-prohibitive with greater spatial resolution. To contrast this, the satellite used here provides free access to its data, making it much more accessible.

While the ViT trains slightly quicker than ResNet50, the overall testing accuracy result is not up to the same level as the CNN mentioned. When training ResNet50 by itself, it achieved a validation accuracy of 86% in 50 epochs, with the validation loss on the final epoch staying at 0.62, and the overall low loss of the final epoch is also reflected in Figure 6. This is a good result that could be used standalone, but the model started to show diminishing returns over the last epochs.

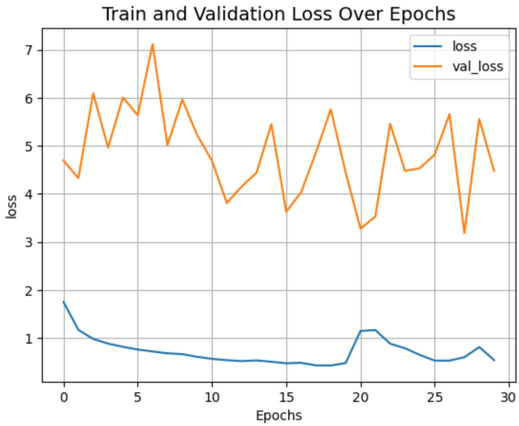

Figure 7. Hybrid model validation loss

4.1 The hybrid model approach

With these results, a hybrid approach to integrating ViTs in image classification tasks alongside CNNs can be a show of strength for both types of models, especially considering the previous limitations of using a ViT by itself. This approach was proposed by Dosovitskiy et al. [1], however, as mentioned by Ngo et al. [5], there are several ways to both training and implementing ViTs with CNNs. For this use case, the ViT is used to replace the final layers of the model ResNet50 to handle the image classification. This approach ensures minimal conflicts between input shapes for the layers used.

Similar to the individual ViT, the overall performance of the hybrid model is very poor. The validation loss shown in Figure 7, and the total accuracy of 12% shows that even a feature extractor like ResNet50 cannot improve the classification aspects of a vision transformer on low-level applications. Furthermore it proves that overfitting still occurs even when coupled with the efficiency of CNNs on small datasets.

5 Conclusion & Future work

Vision transformers are not particularly effective for classifying satellite images when used alone or in combination with convolutional neural networks on small datasets. Although combining vision transformers with convolutional neural networks generally mitigate some of the individual weaknesses of each architecture on larger sets, the overall performance remains ineffective for small datasets. While this raises concerns about the viability of vision transformers in such scenarios, it also presents an opportunity to explore their integration as a complementary component embedded within convolutional neural networks and to investigate further optimization techniques tailored for small datasets.

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Section 3

Applied Mathematics

MILP Model for Scheduling Jobs and Crews with Employee Workload Balancing and Job Lateness

Radu Buzatu

Abstract

This study aims to propose an original and flexible mixed integer linear programming (MILP) model for scheduling jobs and crews. The proposed model minimizes the costs of assigning jobs to crews, employee workload heterogeneity and job lateness.

Keywords: scheduling, crew, workload balancing, lateness, milp

1 Introduction

Scheduling jobs and crews is essential in manufacturing and the service industry. Effective jobs assignment to crews can significantly reduce costs and increase company productivity level.

Paper [1] presents an excellent comparative analysis of mixed integer programming models (MILP) for the classical job shop scheduling problem. Many studies have also been conducted in various areas concerning job scheduling. Here are a few examples. The bus crew scheduling problem is studied in [2]. The manpower scheduling problem in manufacturing is explored in [3]. Such studies propose MILP models and heuristic algorithms to solve the problem and obtain optimal or near-optimal solutions.

This paper presents an original MILP model for scheduling job assignments to crews. The objective function is to minimize the total costs of assigning jobs to crews, employee workload heterogeneity and job lateness.

Regarding the problem, we consider the following assumptions. Each job is assigned to exactly one crew at a time. Each crew can perform many jobs at once, so the number of crew employees participating in work can be at most the total number of crew employees. For each pair of jobs and crews, there is a set of work options associated with the number of crew employees allocated to perform the job. The earliest start and the latest admissible end times are given for each job. Job lateness is allowed, but it is penalized. For each job, some predecessor jobs can be defined. There is a cost of assigning a job to a crew.

2 MILP notations

We consider time discrete, and the entire planning horizon is divided into elementary time intervals of the same duration $d, d > 0$. The index of an elementary time interval is associated with its start time, and it is denoted by t. The start time of each elementary time interval is found by multiplying its time index by d.

Sets and parameters:

J - set of jobs

C - set of crews

$O_{j,c}$ - set of work options of crew c on job j. An option is associated with the number of employees of crew c allocated to perform job j

T - set of time interval indexes of planning horizon, $T = \{0, 1, ..., |T|-1\}$, where 0 and $|T|-1$ are the planning horizon's first and last indexes. Here, 0 and $|T| \cdot d$ are the planning horizon's start and end times.

T_c^{crew} - set of working time intervals of crew c, $T_c^{crew} \subseteq T$

e_j - the earliest admissible time index for starting work on job j

l_j - the latest admissible time index for ending work on job j

n_c^{crew} - total number of employees in crew c

$n_{j,c,o}^{work}$ - number of employees required to perform job j by crew c according to option o

$d^{all}_{j,c,o,t}$ - number of all (working and non-working) time intervals during which work on job j is performed by crew c according to option o when work starts at the time corresponding to time index t

$d^{work}_{j,c,o,t}$ - number of working time intervals during which work on job j is performed by crew c according to option o when work starts at the time corresponding to time index t

$p_{j,j'}$ - predecessor indicator. $p_{j,j'} = 1$ iff job j is an immediate predecessor of job j'

$g_{j,j'}$ - gap time between ending of job j and starting of job j'

$\alpha_{j,c}$ - cost of choosing crew c for performing job j. This cost may express the skill of crew c in performing job j

β_j - lateness cost of job j

γ - cost of employee workload heterogeneity

Decision variables:

$x_{j,c,o,t}$ - binary variable that is equal to 1 iff crew c performs work on job j according to option o and starting at the time corresponding to time index t

y_j - non-negative integer variable indicating lateness of job j

z^{low}, z^{high} - non-negative real variables indicating the lowest and, respectively, the highest employee workload

3 MILP model

The objective (1) of the model shown below is to minimize the costs of assigning jobs to crews, employee workload heterogeneity and job lateness. Constraints (2) ensure that a unique crew, work option (number of employees), and start time are chosen for each job in the final solution. Constraints (3) guarantee that the number of crew employees assigned to the job at any given time interval is at most equal to the total number of crew employees. Constraints (4) require that the start time of any job be greater than the completion time of all its predecessors plus a given gap time. Constraints (5) are used to calculate job lateness. Constraints (6) and (7) are used to calculate the highest and the lowest workload per employee. Finally, constraints (8)-(10) define types of variables.

$$\sum_{j\in J}\sum_{c\in C}\alpha_{j,c}\cdot\sum_{o\in O_{j,c}}\sum_{\substack{t\in T_c^{crew}\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} x_{j,c,o,t}+$$

$$+\sum_{j\in J}\beta_j\cdot y_j+\gamma\cdot(z^{high}-z^{low})\rightarrow min \quad (1)$$

$$\sum_{c\in C}\sum_{o\in O_{j,c}}\sum_{\substack{t\in T_c^{crew}\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} x_{j,c,o,t}=1 \quad \forall j\in J \quad (2)$$

$$\sum_{j\in J}\sum_{o\in O_{j,c}} n^{work}_{j,c,o}\cdot\sum_{\substack{t'\in T_c^{crew}\\ t-d^{all}_{j,c,o,t'}+1\le t'\le t}} x_{j,c,o,t'}\le n_c^{crew} \quad \forall c\in C,\forall t\in T_c^{crew} \quad (3)$$

$$\sum_{c\in C}\sum_{o\in O_{j,c}}\sum_{\substack{t\in T_c^{crew}\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} (t+d^{all}_{j,c,o,t})\cdot x_{j,c,o,t}+g_{j,j'}\le$$

$$\le\sum_{c\in C}\sum_{o\in O_{j',c}}\sum_{\substack{t\in T_c^{crew}\\ e_{j'}\le t\le |T|-d^{all}_{j',c,o,t}}} t\cdot x_{j',c,o,t} \quad \forall j,j'\in J, p_{j,j'}=1 \quad (4)$$

$$\sum_{c\in C}\sum_{o\in O_{j,c}}\sum_{\substack{t\in T_c^{crew}\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} (t+d^{all}_{j,c,o,t})\cdot x_{j,c,o,t}-l_j-1\le y_j \quad \forall j\in J \quad (5)$$

$$\sum_{j\in J}\sum_{o\in O_{j,c}} d^{work}_{j,c,o}\cdot n^{work}_{j,c,o}\cdot\sum_{\substack{t\in T_c^{crew}\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} x_{j,c,o,t}\le n_c^{crew}\cdot z^{high} \quad \forall c\in C \quad (6)$$

$$\sum_{j\in J}\sum_{o\in O_{j,c}} d^{work}_{j,c,o}\cdot n^{work}_{j,c,o}\cdot \sum_{\substack{t\in T^{crew}_c\\ e_j\le t\le |T|-d^{all}_{j,c,o,t}}} x_{j,c,o,t} \ge n^{crew}_c \cdot z^{low} \quad \forall c\in C \quad (7)$$

$$x_{j,c,o,t}\in\{0,1\} \quad \forall j\in J, \forall c\in C, \forall o\in O_{j,c}, \forall t\in T^{crew}_c \quad (8)$$

$$y_j\in\mathbb{N} \quad \forall j\in J \quad (9)$$

$$z^{high}, z^{low}\in\mathbb{R}_+ \quad (10)$$

4 Conclusion

We proposed a precise MILP model for flexible scheduling jobs and crews with employee workload balancing and job lateness. Modern commercial MILP solvers (such as CPLEX and Gurobi) can tackle the jobs and crews scheduling problem in case of small and medium-sized instances. The near-optimal heuristic procedure can be used for large instances of the problem. Our next task is to further develop the model by adding job earliness and extending relationships between jobs.

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Collocation Method for Solving Singular Integral Equations with Discontinuous Coefficients

Maria Capcelea, Titu Capcelea

Abstract

The paper presents an efficient method for solving the Cauchy singular integral equation, which is defined on a closed and smooth contour in the complex plane. The coefficients and the right-hand side of the equation are piecewise continuous functions, numerically defined on a finite set of points along the contour.

Keywords: Singular integral equation, piecewise continuous coefficients, collocation method, B-spline functions.

1 Introduction and problem formulation

Let a closed and smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset \mathbb{C}$, with the point $z = 0 \in \Omega^+$. We consider the points on the contour Γ to be defined by the Riemann function $z = \psi(w)$, which maps the circle $\Gamma_0 = \{w \in \mathbb{C} : |w| = 1\}$ onto the contour Γ. Let's consider the Cauchy singular integral equation defined on the contour Γ

$$c(t)\varphi(t) + d(t)(S\varphi)(t) + (K\varphi)(t) = f(t), \quad t \in \Gamma, \tag{1}$$

where $(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau$ is the Cauchy singular integral, and $(K\varphi)(t) = \frac{1}{2\pi i}\int_\Gamma h(t,\tau)\varphi(\tau)\,d\tau$ is the regular part, with the kernel $h \in C(\Gamma \times \Gamma)$. We assume that the coefficients c and d of the equation, the right-hand side f, and the solution φ are piecewise continuous functions.

Let $\{t_j\}_{j=1}^{n_B}$, $t_j = \psi(w_j)$, $w_j = e^{i\theta_j}$, $\theta_j = 2\pi(j-1)/n_B$, $j = 1,...,n_B$, be the set of distinct points of the contour Γ, where the values of the right-hand side f and the coefficients c and d are defined. Additionally, we assume that the values of the functions f, c and d are known at the discontinuity points t_r^d, $r = 1,...,n_{pd}$, on the contour Γ.

We aim to develop a numerical method that determines a sequence of approximations φ_n, which converge uniformly to the solution φ of equation (1) everywhere on Γ, except for a finite number of discontinuity points of φ. To approximate the solution of equation (1), we apply a modification of the collocation method, where the solution is sought as a linear combination of B-spline functions and Heaviside step functions, resulting in a piecewise continuous approximation function.

2 The numerical method

The method we propose for approximating the solution φ of equation (1) uses the concept of B-spline functions of order m, which are defined at the points t_j, $j = 1,...,n_B$, by the recursive formula $B_{m,j}(t) = \frac{m}{m-1}\left(\frac{t-t_j^B}{t_{j+m}^B-t_j^B}B_{m-1,j}(t) + \frac{t_{j+m}^B-t}{t_{j+m}^B-t_j^B}B_{m-1,j+1}(t)\right)$. The set of nodes $\left\{t_j^B\right\}_{j=1}^{n_B+m}$ satisfies the condition $t_j^B = t_j$, $j = 1,...,n_B$, $t_{n_B+1}^B = t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$. It is known that any continuous function on Γ can be uniformly approximated by a linear combination of B-spline functions. When linear combinations of B-splines are used to approximate piecewise continuous functions, the approximation error decreases as the number of spline knots increases, even in the vicinity of discontinuity points [1].

With respect to the points of discontinuity t_r^d, $r = 1,...,n_{pd}$, we define the Heaviside step function H on the contour Γ:

$$H\left(t - t_s^d\right) = \begin{cases} 0 & if\ t \in \left(\Gamma_1 \setminus \left\{t_1^B\right\}\right) \cup ... \cup \Gamma_{q-1} \cup arc\left[t_q^B, t_s^d\right) \\ 1 & if\ t \in arc\left[t_s^d, t_{q+1}^B\right) \cup \Gamma_{q+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

where $\Gamma_q = arc\left[t_q^B, t_{q+1}^B\right]$, $t_s^d \in \Gamma_q$.

Taking into account that the solution φ of equation (1) exhibits jump discontinuities on the contour Γ, we seek to approximate φ in the form [1]

$$\varphi_{n_B}^{H}(t) = \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t) + \sum_{s=1}^{n_{pd}} \beta_s H\left(t - t_s^d\right),$$

where the coefficients $\alpha_k \in \mathbb{C},\ k = 1, ..., n_B$, and $\beta_s \in \mathbb{C},\ s = 1, ..., n_{pd}$, are determined by imposing $n = n_B + n_{pd}$ interpolation conditions

$$c\left(t_j^C\right)\varphi_{n_B}^{H}\left(t_j^C\right) + d\left(t_j^C\right)\left(S\varphi_{n_B}^{H}\right)\left(t_j^C\right) + \left(K\varphi_{n_B}^{H}\right)\left(t_j^C\right) = f\left(t_j^C\right). \quad (2)$$

In relation (2), the first n_B interpolation points $t_j^C,\ j = 1, ..., n_B$, are the nodes $t_j^B = t_j,\ j = 1, ..., n_B$, and the remaining n_{pd} points $t_j^C,\ j = n_B + 1, ..., n$, are the discontinuity points $t_s^d,\ s = 1, ..., n_{pd}$, of the function f (and also of c and d). If among the interpolation points $t_j^C,\ j = 1, ..., n_B$, there are discontinuity points $t_s^d = \psi\left(e^{i\theta_s^d}\right)$ of the function f on Γ, then instead of them, for sufficiently small $\varepsilon_2 > 0$, we consider points $\tilde{t}_s^d = \psi\left(e^{i\left(\theta_s^d - \varepsilon_2\right)}\right)$ such that $f\left(\tilde{t}_s^d\right) = f\left(t_s^d\right)$ (also for values of c and d). The Riemann integrals $\left(K\varphi_{n_B}^{H}\right)\left(t_j^C\right)$ are aproximated using the generalized trapezoidal rule, which is also applicable to functions with complex values.

As the interpolation points $t_j^C,\ j = 1, ..., n$, form a subdivision covering the contour Γ, if $t_j^C \in arc\left[t_{k+r-1}^B, t_{k+r}^B\right]$, the integral $\left(S\varphi_{n_B}^{H}\right)\left(t_j^C\right)$ will exhibit a singularity of type $1\Big/\left(\tau - t_j^C\right)$. In such cases, we interpret $\left(S\varphi_{n_B}^{H}\right)\left(t_j^C\right)$ as the Cauchy principal value and require an efficient method for approximating this integral.

Let θ_j^C be the polar angle corresponding to the point $t_j^C = \psi\left(e^{i\theta_j^C}\right) \in \Gamma$. The singular integral $\left(S\varphi_{n_B}^{H}\right)\left(t_j^C\right)$ can be expressed as $I = p.v.\frac{1}{\pi i}\int_c^d g(\theta)\, d\theta$, where $g(\theta) = f(\theta) / (\varsigma(\theta) - \varsigma(\theta_0)),\ \theta_0 = \theta_j^C$. Using the notation $a = min\{\theta_0 - c, d - \theta_0\}$ $(a > 0)$, the integral I can be represented as $I = (1/\pi i)(I_1 + I_2)$, where $I_1 = p.v.\int_{\theta_0 - a}^{\theta_0 + a} g(\theta)\, d\theta$,

$I_2 = \begin{cases} \int_{\theta_0+a}^{d} g(\theta)\, d\theta \;\; if \;\; a = \theta_0 - c \\ \int_{c}^{\theta_0-a} g(\theta)\, d\theta \;\; if \;\; a = d - \theta_0 \end{cases}$. The value of I_2 represents a Riemann integral, while the integral I_1 is interpreted as a generalization of the Cauchy principal value. The integral I_2 will be computed using the generalized trapezoidal rule, and for the computation of I_1, an analogous procedure to that outlined in [2] can be applied.

3 Conclusion

This paper presents a new method for solving Cauchy singular integral equations with piecewise continuous coefficients defined on a smooth and closed contour in the complex plane. The method constructs a sequence of approximations that converge almost uniformly to the exact solution of the integral equation.

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Experimental Closed-Loop Identification of the Third Order Inertial Systems with Time Delay

Irina Cojuhari

Abstract

In this paper, it is analysed the problem of mathematical identification in the closed-loop and it is proposed an approach for experimental identification, based on the undamped step response of the closed-loop control system. According to the proposed procedure, the dynamics of the control object is proposed to be approximated with transfer function with inertia third order and time delay. The obtained results are verified by computer simulation in MATLAB software package.

Keywords: Experimental closed-loop identification, mathematical modelling, inertial systems, transfer function, time delay.

1 Introduction

Mathematical modelling of a control object in control engineering involves obtaining a mathematical representation of a physical process to describe its behaviour in response to various inputs [6]-[7]. This process is essential for designing effective control strategies and begins with a thorough understanding and description of the system. Identifying the mathematical model that approximates the dynamic of the control object can lead to a qualitative understanding of the system's behavior [1].

System identification can be realized in the open-loop system and closed-loop feedback system [4]. Open-loop identification methods are simple in use and these methods are applied, if the process dynamics can be linearized around a nominal operating point, and the process is characterised by the non-varied in time parameters [2]. But for case of using the auto-tuning controllers, it is necessary to obtain the mathematical model of the control object in the feedback closed-loop system, which involves using the closed-loop identification methods [3].

In this paper, it is proposed the procedure for closed-loop identification of the third order inertial system with time delay based on the undamped transient response of the closed-loop system.

2 Algorithm of closed-loop identification of the undamped third order inertial system with time delay

The block scheme of the automatic control system is presented in Fig. 1.

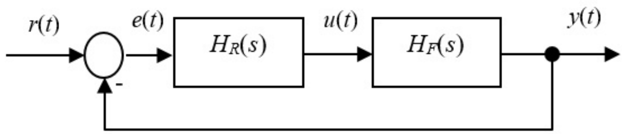

Figure 1. Structural scheme of the automatic control system

It is proposed, that the control object to be approximated with the following transfer function:

$$H_F(s) = \frac{ke^{-\tau s}}{(T_1+1)(T_2+1)(T_3+1)} = \frac{ke^{-\tau s}}{a_0 s^3 + a_1 s^2 + a_2 s + a_3}, \quad (1)$$

where k - transfer coefficient, τ - time delay, T_1, T_2, T_3 - time constants, $a_0 = T_1T_2T_3$, $a_1 = T_1T_2 + T_1T_3 + T_2T_3$, $a_2 = T_1 + T_2 + T_3$, $a_3 = 1$.

For model estimation, it is proposed to be used P controller with transfer function:

$$H_R(s) = k_p, \quad (2)$$

where k_p is a proportional tuning parameter.

From the transfer function (1), the unknown coefficients a_0, a_1, a_2, a_3 are proposed to be calculated based on the values of the k_{cr} – proportional critic transfer coefficient of the P controller, when system achieves undamped step response and T_{cr} – the period of oscillations. To achieve this state of the system, it is considered to be given the closed-loop system with P controller (Fig. 1) and the proportional tuning parameter $k_p > 0$ is varied, until the system achieves the undamped step response presented in Fig. 2.

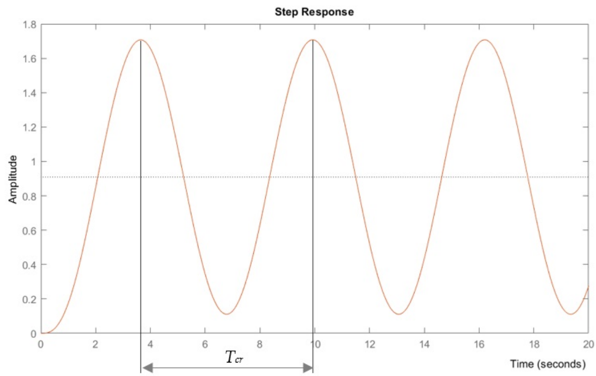

Figure 2. Undamped step response of the closed-loop system

If the values of period oscillations T_{cr} are known, then the value of natural frequency is calculated by [5]:

$$\omega_n = \frac{2\pi}{T_{cr}}. \tag{3}$$

The closed-loop characteristic equation of the system with P controller is the following:

$$A(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 + k k_{cr} e^{-\tau s}. \tag{4}$$

It is proposed the following substitution of the Laplace transform:

$$s = \omega_n j. \tag{5}$$

Based on Eq. (5), the characteristic equation (4) will become:

$$A(\omega_n j) = (-a_1\omega_n^2 + a_3 + k_{cr}kcos(\tau\omega)) + j(-a_0\omega_n^3 + a_2\omega - kk_{cr}sin(\tau\omega)) =$$

$$= P(\omega) + jQ(\omega). \tag{6}$$

Next, it was set the real and imaginary part with zero, denoted as $P(\omega) = 0$ and $Q(\omega) = 0$; based on this equaling, there are obtained the following expressions for calculation the model parameters a_1, a_2:

$$a_1 = \frac{a_3 + k_{cr}kcos(\tau\omega_n)}{\omega_n^2}; a_2 = \frac{a_0\omega_n^3 + kk_{cr}sin(\tau\omega_n)}{\omega_n}; a_3 = 1. \tag{7}$$

The coefficient a_0 remains unknown and for the calculation of this coefficient it is used the maximum stability degree criterion [7]. According to this criterion, the maximum displacement of the dominant poles of closed loop system to the imaginary axe in the left complex half plane is denoted by the η_{max} that is equal to:

$$\eta_{max} = \frac{a_1}{(n+1)a_0}, \tag{8}$$

where n is order of the characteristic equation of the closed-loop system.

And it is known that the maximum displacement is known as maximum stability degree of the system, denoted by J.

For the case, when steady state error will be $\epsilon_{st} = 0.02$, the settling time is equal to:

$$t_s \approx \frac{4}{J}. \tag{9}$$

By the variation of the value of proportional tuning parameter $k_p > 0$, it is possible to achieve the critically damped step response. And according to this response, there can be calculated the value of settling time -t_s. If this value is known, based on Eq.(8)-(9), it is obtained an expression for calculation a_0:

$$a_0 = \frac{a_1 t_s}{20}. \tag{10}$$

3 Study Case and Computer Simulation

For the verification of the proposed algorithm of experimental identification, it was supposed that the control object to be described by the following tranfer function:

$$H_F(s) = \frac{e^{-20s}}{150s^3 + 95s^2 + 18s + 1}. \tag{11}$$

The given system (11) was simulated in MATLAB with P controller, where the proportional tuning parameter k_p was varied until it was achieved the undamped transient response of the closed loop system for $k_{cr} = 1.53$.

The obtained undamped step response is presented in Fig. 3, from which there are calculated the period of oscillation $T_{cr} = 70.95s$. The value of natural frequency was determinated by the expresion (3): $\omega_n = 0.0885$.

Next, according to Eq.(7), it was calculated the a_1 parameter of the identified transfer function: $a_1 = 88.774$.

Further, the proportional tuning parameter $k_p > 0$ was varied till it achieved the critically damped step response. And according to this response, it was calculated the value of settling time $t_s = 35$ sec. If this value is known, based on Eq.(10), it was calculated a_0:

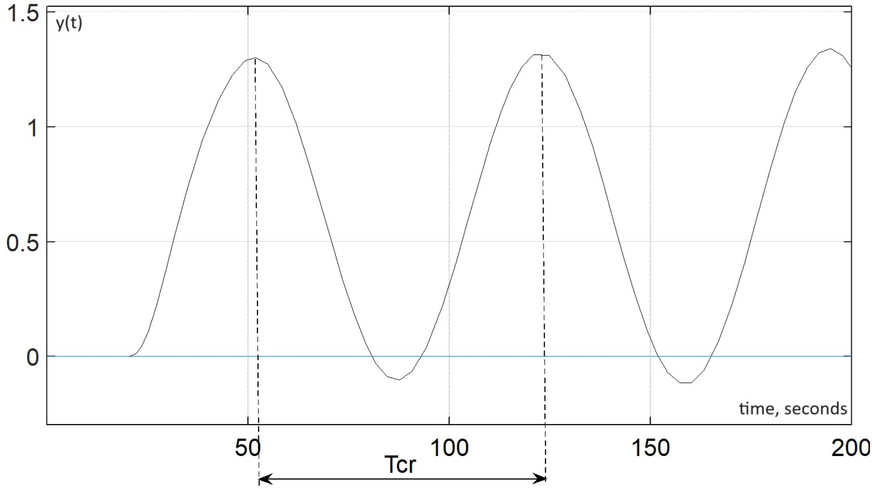

Figure 3. Underdamped step response of the closed-loop system

$$a_0 = \frac{a_1 t_s}{20} = \frac{88.774 * 35}{20} = 155.355. \quad (12)$$

If the value of a_0 is known, then it can be calculated the value of a_2:

$$a_2 = \frac{a_0 \omega_n^3 + k k_{cr} sin(\tau \omega_n)}{\omega_n} = 18.1695. \quad (13)$$

According to these calculations, the identified transfer function is the following:

$$H_F(s) = \frac{e^{-20s}}{156.16s^3 + 88.774s^2 + 18.169s + 1}. \quad (14)$$

In Fig. 4, it is presented the comparation between original step response of the open loop system described by the transfer function (11) – curve 1 and the identified transfer function (14) – curve 2.

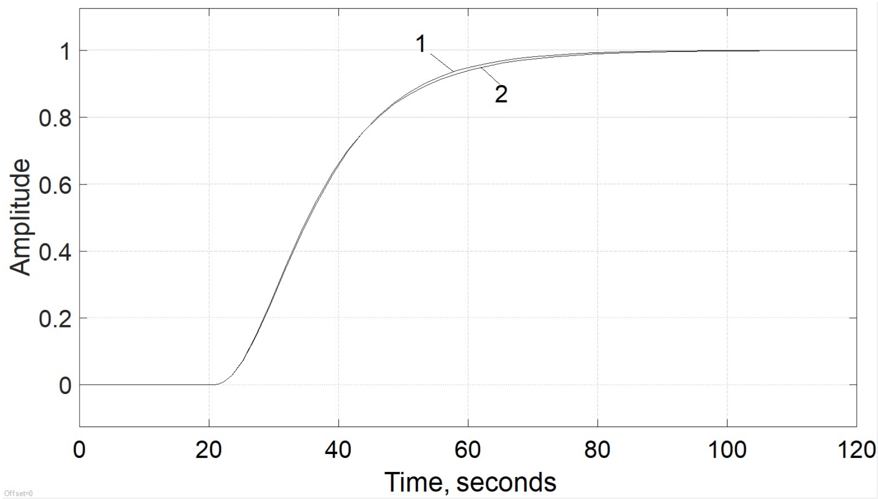

Figure 4. Comparation of the system step responses in the open loop: 1-original transfer function (11); 2-identified transfer function (14)

From Fig. 4, it can be observed that the proposed methodology of model estimation in the closed-loop offers so good results in model estimation.

4 Conclusion

In this paper, it was proposed an approach for experimental identification in the closed-loop of the third order inertial system with time delay. The proposed algorithm supposes to be achieved the limit of stability of the closed-loop system, and based on the parameters, that are obtained from the undamped response of the system, there are presented some simple expressions for calculation the parameters of the model. The closed-loop identification was verified by computer simulation and the obtained results demonstrated that the identification procedure in the closed-loop offers good results in process of estimation the mathematical model in the closed-loop, and this method can be used to design the auto-tuning controllers.

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Semi-Markovian Networks

Iulia Damian

Abstract

In stochastic networks, a series of new problems appear, because the output flow of messages from one node forms the input flow to another node (or other nodes), which is not Poisson, and the serving of messages in the nodes of these networks is no longer exponential type. Such networks are called semi-Markovian networks and, therefore, the mathematical analysis of semi-Markovian networks is complicated and requires the application of some modern departments of mathematics.

Keywords: queuing system, networks, exponential distribution, semi-Markovian networks, averaging method, diffusion approximation method.

1 Introduction

The theory of semimarkovian random evolution has a wide and diversified application, for example, in the theory of inventory, consumption, and transport; in the theory of stochastic differential equations; in the analysis of real stochastic systems that appear when solving problems applied in physics, biology, ecology, and economics. The theory of stochastic networks emerged from the need to exploit informational and numerical systems, economic networks, and telecommunication networks. The processes, which describe the activity of these stochastic networks, can have a rather complex and large structure. The analysis and optimization of complex systems are multiparametric problems, being an important step in solving optimization problems, for complex systems under conditions of uncertainty and the risk of the primary mathematical model are usually stochastic. This, for example, appears in the theory of information processing networks. The significant development of the theory of the evolution of stochastic systems is described in the works of Korolyuk V. S. [1,2], Korolyuk V. S. and Svischuka A. V. [3], Korolyuk V. S. and Korolyuk V. V. [4], Korolyuk V. S. and Turbin A.F., Skorokhod A.V., Ghihman I. I. and Skorokhod A.V., and others. The vertiginous development of networks, including wireless information transport networks, have presented a series of pressing problems. Among these problems, there is the development and analysis of mathematical models of these processes that take place in real contemporary systems and networks. In recent years, in the attention of specialists in the field of operational research and applied probabilities, there are the so-called stochastic serving networks. Here a series of new problems arise because in stochastic networks, the output flow of messages from one node forms the input flow to another node (or other nodes), which is not of Poisson type. At the same time, the serving of messages in the nodes of these networks is no longer governed by the exponential distribution. Such networks are called semi-Markov or semi-Markovian networks. That is why the mathematical analysis of semi-Markovian networks is complicated and requires the application of some modern departments of mathematics. Most queuing systems in real life are usually networks. An example is the packet-switched communication system, road networks, etc. However, a queuing network system can be viewed as a simple collection of multiple strings with a single node. Sometimes a queuing network is decomposed into multiple single-node strings as an approximation for analysis purposes. This is due to the fact that, analyzing waiting networks or even tandem networks, they could be very involved, except for simple unrealistic cases. Korolyuk V. S. was concerned with waiting networks. Thus, waiting networks were also investigated in Moldova by Gutuleac E. [5].

2 Queuing systems

A queuing system is the system in which the elements end up being processed. The terms “element” and “processing” are generic terms.

Examples:

1) An “element” could refer to customers. The elements could be customers arriving at a system, such as a bank, to receive services (to be “processed”).

2) Is in the manufacturing process where an item might be a partially completed part that needs to be machined and so is sent to a station where it is properly machined when possible.

3) The transport unit passing through an intersection is an element that needs to be served, and the service is provided by the intersection in the form of turning on the green light to go through it. In general, in a queuing system, there is a physical or virtual location, sometimes moving, where items end up being processed. If the processor is available, the element can be processed immediately, otherwise it must wait, if the processor is busy. Almost all of us have experienced queuing systems directly or indirectly: directly, when we sit and wait in line, or indirectly, through elements of our own standing in a waiting line, such as when a job waiting to be printed in the printer queue's queue space, or a packet belonging to a person waiting at a router node for processing. In any case, we all want the waiting time to be minimal and also not to be returned from the system due to the unavailability of waiting space. Of course, we all know that if this management system can provide “enough” resources to process the items, then the possibilities of delays or denial of service will be small. However, the system manager providing the service is limited in how many resources can be provided because they cost money.

3 The queuing system $[SM|M|1|\infty]^N$ and $SM^{\varepsilon}|M|\infty$

The queuing system $[SM|M|1|\infty]^N$ means:

- the input flow is described by a semi-Markov process

- the servicetime is distributed exponentially

- there are N servers connected by a route probability matrix

These queuing networks are considered with a semi-Markov flow.

The queuing system $SM^{\varepsilon}|M|\infty$ means:

- arrival times are the jumps times of a semi-Markov process on the standard phase space (E, ε)

- the service time is exponentially distributed with one server, while the end of service times are the jumps of Markov process when the queuing system is busy.

The most effective mathematical methods for simplifying waiting systems are: the averaging method and the diffusion approximation method. The averaging method allows to obtain the determined simplified model of the waiting system [6,7]. Since the general mediation model is not fully primary, it is necessary to use the diffusion approximation to evaluate the fluctuations of the initial model. Using the diffusion approximation, we can describe the fluctuations of the initial model against the mean. Solving the optimization problem for the simplified mediation model is much easier than optimizing the original model. Minimizing fluctuations, managing initial network parameters is one of the important practical issues.

4 Conclusion

The theory of queuing systems is stochastic in nature. Some components of the system, such as the number of serving nodes, the queue waiting to be served, the intensity of the flow, can be both deterministic and random. Economic examples: some companies satisfy the needs of the population, which can be: shops, hospitals, repair shops, laundries, computer clubs, driving licenses, etc. The company generates revenue that satisfies the customer's needs. Profits depend on constant and variable components. The constant component includes the production cost of assets (buildings, equipment), rent, payment for utilities, minimum wage, taxes, etc. The variable component depends on the number of actual requests that have been fulfilled in a certain period of time.

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Strong Edge-Colorings of Some Lexicographic Products of Graphs

Aram Drambyan

Abstract

The edge-coloring of a graph G is strong if edges at distance 0 or 1 receive different colors. The minimum number of colors required for a strong edge-coloring of G is called strong chromatic index of G and denoted by $\chi'_s(G)$. In this paper, we determine the exact value of the strong chromatic index of lexicographic products of graphs G and H when $\Delta(G) \leq 2$ and H is an arbitrary graph, and provide an upper bound on $\chi'_s(G[H])$ when $\Delta(G) = 3$ and H is an arbitrary graph.

Keywords: edge-coloring, strong edge-coloring, strong chromatic index, lexicographic product of graphs.

1 Introduction

We consider only finite and simple graphs. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of G by $\Delta(G)$. The *distance between two edges* $e, e' \in E(G)$ in a graph G is the length of the shortest path between endpoints of e and e'. A strong *edge-coloring* of a graph G is a mapping $\phi : E(G) \to \mathbb{N}$, where the edges that are at distance 0 or 1 from each other receive different colors. *The strong chromatic index* of a graph G is the minimum number of colors required for a strong edge-coloring of G, and is denoted with $\chi'_s(G)$. The problem of determining a strong chromatic index of a graph was suggested by Fouquet and Jolivet in 1983 [5].

In 1985, Erdős and Nešetřil conjectured that for any graph G with maximum degree $\Delta(G)$, the strong chromatic index is at most $\frac{5}{4}\Delta(G)$ (see [1]). Later, Faudree et al. conjectured that for any bipartite graph G with maximum degree $\Delta(G)$, $\chi'_s(G) \leq \Delta(G)^2$ [1]. The conjecture proposed by Erdős and Nešetřil was proved for graphs G with $\Delta(G) = 3$ by Andersen and independently by Horák et al. in [2, 3]. For graphs G with $\Delta(G) = 4$, the currently best known upper bound on $\chi'_s(G)$ is 21 and it was proved in [6]. In [4], Togni investigated strong edge-colorings of Cartesian, Kronecker and strong products of graphs. In this paper, we bound or determine the strong chromatic index of lexicographic products of graphs G and H when $\Delta(G) \leq 3$ and H is an arbitrary graph.

2 Main Results

Let G and H be two graphs.

Definition 1 . *The lexicographic product of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which $x = (u_1, v_1)$ is adjacent to $y = (u_2, v_2)$ whenever u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent to v_2, and is denoted by $G[H]$.*

Definition 2 . *For any $v \in V(G)$ and H, we denote by $v\,H$, the subgraph of $G[H]$ induced by vertices $\{(v, u) \mid u \in V(H)\}$.*

Definition 3 . *For any $v_1, v_2 \in V(G)$ and H, we denote by $[v_1, v_2]H$, the subgraph of $G[H]$, where $V([v_1, v_2]H) = \{(v, u) \mid v \in \{v_1, v_2\}, u \in V(H)\}$ and $E([v_1, v_2]H) = \{(v_1, u_1)(v_2, u_2) \mid u_1, u_2 \in V(H)\}$. By the definition of the lexicographic product of graphs G and H, it follows that $[v_1, v_2]H$ is isomorphic to the complete bipartite graph.*

We begin our considerations with strong edge-colorings of the lexicographic products of graphs G and H, where $\Delta(G) \leq 2$. Since for a graph G, $\Delta(G) \leq 2$, each component of G is either a path or a cycle.

Theorem 1. *For any graph H and $n \geq 4$,*

$$\chi'_s(P_n[H]) = 2\chi'_s(H) + 3|V(H)|^2$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$. First, let us consider the edges incident to vertices of subgraphs v_2H and v_3H. All the edges from $[v_1, v_2]H$, $[v_2, v_3]H$, and $[v_3, v_4]H$ subgraphs should receive different colors as they are at distances 0 or 1 from each other. At the same time, edges from v_2H and v_3H subgraphs, are also at distances 0 or 1 from edges in $[v_1, v_2]H$, $[v_2, v_3]H$, $[v_3, v_4]H$, and each other. Using these constraints, we get $\chi'_s(P_n[H]) \geq 2\chi'_s(H)+3|V(H)|^2$. To construct a strong edge-coloring of the graph $P_n[H]$, which uses exactly $2\chi'_s(H) + 3|V(H)|^2$ colors, we will start from the coloring of subgraphs vH. For each i $(1 \leq i \leq n)$, we assign v_iH subgraph, to group one if i is even and to group two if i is odd. Subgraphs in each group can be colored using the same set of colors as the edges between subgraphs that are at distances at least 2 from each other. Using the fact that each subgraph can be colored with $\chi'_s(H)$ colors, we can color all of them using $2\chi'_s(H)$ colors. Now we will color $[v_i, v_{i+1}]H$ subgraphs $(1 \leq i \leq n-1)$ using $3|V(H)|^2$ colors. Clearly, for the coloring of each $[v_i, v_{i+1}]H$ subgraph, we need $|E([v_i, v_{i+1}]H) = |V(H)|^2$ colors as it is isomorphic to the complete bipartite graph. Subgraphs $[v_i, v_{i+1}]H$ preserve the same distance properties as edges of P_n. In other words, subgraphs $[v_i, v_{i+1}]H$, that either share vertices or edges should be colored using different colors. From the fact that $\chi'_s(P_n) = 3$, we get that we need $3|V(H)|^2$ new colors to complete the strong edge-coloring of $P_n[H]$. Thus, our coloring uses $2\chi'_s(H) + 3|V(H)|^2$ colors, which matches with a lower bound. □

Using a more careful analysis, we prove that the following results hold.

Theorem 2. *For any graph H and $n \geq 5$,*

1. *If $n = 5$, then $\chi'_s(C_n(H)) = 5|V(H)|^2$.*
2. *If $n = 6k$, then $\chi'_s(C_n(H)) = 2\chi'_s(H) + 3|V(H)|^2$.*
3. *If $n = 6k+3$, then $\chi'_s(C_n(H)) \leq 3\chi'_s(H) + 3|V(H)|^2$.*

4. If $n = 6k+1, 6k+2, 6k+4, 6k+5,$ *then*

$$\chi'_s(C_n(H)) \leq 2\chi'_s(H) + 4|V(H)|^2.$$

Theorem 3. *For any graph* G *with* $\Delta(G) = 3$ *and any graph* H*,*

$$\chi'_s(G[H]) \leq 5\chi'_s(H) + 10|V(H)|^2.$$

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Algorithm to Minimize the Worst-Case Regret Function

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Abstract

This paper examines mathematical model of decision-making under conditions of uncertainty, when Savage's regret function is used as the objective function. Numerical algorithm for solving the corresponding problems are proposed, based on the implementation of parallel minimization of indicators for each state of nature and the indicator of greatest regret. The algorithms are developed based on the generalized gradient projection method.

Keywords: uncertainty, regret function, optimization, numerical algorithm.

Introduction

When making decisions under conditions of uncertainty, and sometimes under conditions of risk, it is necessary to take into account the discrepancy between the actual result obtained (win or loss) and the "allowed" one – the best option that corresponded to the case if the current state of nature was guessed. This disagreement is measured by a value called "regret". The concept of regret, namely in matrix form, was first introduced by Savage in [1].

We will follow work [2], which defines the Savage function of the form

$$R_s\left(u\right) = \max_{i \in I_m} (\bar{r}\left(u, q_i\right)), \tag{1}$$

where $\bar{r}(u, q_i) = (r(u, q_i) - r_i^*)$, $r_i^* = r\left(u_{(i)}^*, q_i\right) = \underset{u \in D}{min}\ r(u, q_i)$.

Here $r(u, q_i)$ is the cost function for the case when the decision is made $u \in D$, and nature is determined by the state $q_i \in Q = \{q_1, q_2, \ldots, q_m\}$, $i \in I_m = \{i : i = 1, 2, \ldots, m\}$.

Let us assume, on a compact convex set U of n-dimensional Euclidean space $\mathrm{E^n}$, all functions $r(u, q_i)$, $i \in I_m$ and $F_j(u), j \in J_t = \{1, 2, ..., t\}$ are continuous and convex. Obviously, the functions also have the same properties $\bar{r}(u, q_i)$ and $R_s(u)$. Let subgradients $r'_u(u, q_i)$ of functions $r(u, q_i)$, $i \in I_m$, and subgradients $g_j(u)$ of functions $F_j(u)$, $j \in J_t$, exist for all boundary points of the set U, and they are uniformly bounded in U. The set $D = \{u \in U : F_j(u) \leq 0, j \in J_t\}$.

To minimize the function of worst-case regrets $R_s(u)$ on set D, parallel implementation of **m+1** iterative processes will be considered. First ***m*** of them is aimed at obtaining "sufficient" estimates for unknown values of r_i^*, $i \in I_m$; and process number **(m+1)** –- to determine approximations of the value $R_S^* = \underset{u \in D}{min} R_S(u)$. It is assumed that the set U has a relatively simple structure, in the sense that for an arbitrary point $u \in E^n$, one can precisely determine its projection onto U. We denote this projection by $\Pi_U(u)$.

If we define a function $F(u) = \underset{j \in J_t}{max} F_j(u)$, then the mathematical model in the new formulation is presented as follows:

$$\begin{cases} R_s(u) \to min \\ F(u) \leq 0 \\ u \in U \end{cases}.$$

Thus, the specified **m+1** processes are implemented according to the scheme:

$$\begin{cases} u_{(i)}^{k+1} = \Pi_U\left(u_{(i)}^k - h_{(i)k} \cdot \eta_{(i)}^k\right), i \in I_m \\ u^{k+1} = \Pi_U\left(u^k - h_k \cdot \eta^k\right). \end{cases} \quad (2)$$

$$\text{Here: } \eta_{(i)}^{k} = \begin{cases} \frac{r_u'(u_{(i)}^k, q_i)}{\left\| r_u'(u_{(i)}^k, q_i) \right\|} \text{ if } F\left(u_{(i)}^k\right) \le \bar{\varepsilon}_k \text{ and } r_u'(u_{(i)}^k, q_i) \ne 0 \\ \frac{g(u_{(i)}^k)}{\left\| g(u_{(i)}^k) \right\|} \text{ if } F(u_{(i)}^k) > \bar{\varepsilon}_k \text{ and } g\left(u_{(i)}^k\right) \ne 0 \\ 0, \text{ in other cases, where } i \in I_m \end{cases} \tag{3}$$

$$\eta^k = \begin{cases} \frac{g_s^k(u^k)}{\| g_s^k(u^k) \|}, \text{ if } F(u^k) \le \bar{\varepsilon}_k \text{ and } g_s^k(u^k) \ne 0, \\ \frac{g(u^k)}{\| g(u^k) \|}, \text{ if } F\left(u^k\right) > \bar{\varepsilon}_k \text{ and } g\left(u^k\right) \ne 0, \\ 0, \text{ in other cases} \end{cases}, \tag{4}$$

where $g_s^k\left(u^k\right)$ – arbitrary subgradient function

$$R_S^k(u) = \max_{i \in I_m} \left[r(u, q_i) - r\left(u_{(i)}^k, q_i\right) \right] \tag{5}$$

and $g(u^k)$ – arbitrary subgradient function $F(u)$, calculated at point $u = u^k$.

Let $\mathbf{U}^*$ – set points minimum functions Savage in the $\mathbf{D}$ area. If it is additionally assumed that subgradients $g_s^k(u)$ and $g(u)$ of functions $R_s^k(u)$ and $F(u)$ are uniformly bounded on $\mathbf{U}$, then the following theorem holds.

Theorem. *Let for number sequences $h_{(i)k}, h_k$ and $\bar{\varepsilon}_k$, the following conditions are met:*

$$h_{(i)k}, h_k \ge 0;\ h_{(i)k}, h_k \to 0;\ \bar{\varepsilon}_k > 0;\ \bar{\varepsilon}_k \to 0;\ \frac{h_{(i)k}}{\bar{\varepsilon}_k};\ \frac{h_k}{\bar{\varepsilon}_k} \to 0,$$

$$\sum_{k=0}^{\infty} h_{(i)k} \bar{\varepsilon}_k = \infty, \sum_{k=0}^{\infty} h_k \bar{\varepsilon}_k = \infty. \tag{6}$$

When the above conditions (2) - (6) are met:

$$\lim_{k\to\infty} \min_{u^*\in U^*} \left\| u^k - u^* \right\| = 0, \lim_{k\to\infty} R_S^k \left(u^k \right) = R_S^*,$$

where R_S^* – minimum function value (1), and $R_S^k(u)$ is determined according to (5).

Conclusions. Suggested algorithm can apply in decision making under uncertain conditions when nature could be in a finite, comparatively small number states, and the decision maker chooses the minimum of worst-case regrets as an indicator.

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Transportation Problems, the Braess Paradox and Network Coordination

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Abstract

Traveling through the transport network or sending information packets via the Internet is implicitly based on game-theoretic considerations: a specific decision-maker (DM), choosing his or her route, takes into account the probability of congestion depending on all DMs, that is, other routes. Based on similar considerations, it is possible to develop models for network traffic. These models explain some paradoxical observations where increasing the capacity of a given network can slow down its traffic under certain circumstances.

Keywords: directed graph, route, subroute, travel time, social optimum, Nash equilibrium.

1 Introduction

Let us develop a model of the transport network and investigate how it reacts to traffic jams [1]. Let us represent the transport network as a directed graph, where the edges are highways, and the nodes are exits, where the DM can leave a certain highway and enter another. Let us assume that each DM wants to travel from node A to node B [2]. It should be taken into account the large number of other DMs to travel from A to B during rush hour [3]. Each edge is characterized by a certain travel time, depending on the traffic on this edge.

2 The Problem Example

For concreteness, let us assume that 4000 cars (corresponding DMs) get from A to B by routes 1, 2. The equilibrium traffic is 2000 cars on each route with a travel time of 65 minutes. Let there be node C on route 1 and node D on route 2 , and the trip on the subroute $C-B$ takes 45 minutes, as does the trip on the subroute $A-D$ these subroutes are insensitive to congestion, and the travel time on each such subroute does not depend on the number of cars traveling on it.

On the other hand, suppose that the travel time on subroute $A-C$ or subroute $D-B$ is $x(A-C)/100$ minutes and $x(D-B)/100$ minutes, respectively, where $x(A-C)$ and $x(D-B)$ are the numbers of cars traveling on subroute $A-C$ and $D-B$, respectively. Then at *x(A−C) = 2000 = x(D−B)* the total travel time along the route $A-C-B$ is 65 minutes. If $x(A-C) = 4000$, then the total travel time along the route $A-C-B$ is 85 minutes; similarly, the total travel time on the route $A-D-B$ is 85 minutes at $x(D-B) = 4000$.

In the language of game theory, here the DMs (drivers) are players, and the strategy of each of them is to choose a route $A-C-B$ or $A-D-B$.

We can expect the formation of equilibrium traffic, say, a Nash equilibrium (a list of 4000 strategies, where the strategy of each player $i = 1, \dots, 4000$ is the best response of that player to the listed strategies of all other players $1, \dots, i-1, i+1, \dots, 4000$).

All Nash equilibria have an equal balance because at $x(A-C) > 2000$, $x(A-D) = 4000 - x(A-C) < 2000$ the total travel time along the route $A-C-B$ exceeds the total travel time along the route $A-D-B$, and therefore, the driver on the route $A-C-B$ has an incentive to switch to another route $A-D-B$.

The updated network corresponds to another unique Nash equilibrium with a travel time of 80 minutes from A to B.

In the state of equilibrium, each driver uses a route $A-C-D-B$ corresponding to the total travel time 80 minutes at $x(A-C) = 4000 = x(D-B)$: if the driver uses $A-C-B$ instead of $A-C-D-B$, the total travel time is 85 (minutes); if the driver instead of $A-C-D-B$ uses $A-D-B$, then the total trip time is equal to 85 (minutes). The creation of edge $C-D$ mentioned effectively made the corresponding subroute $C-D$ the dominant strategy for each driver.

3 The Braess Paradox

Dietrich Braess first demonstrated an example where increasing the resources of a transport network does not guarantee an increase in its equilibrium performance, and with certain combinations of parameters it reduces its performance [4]. This phenomenon is called the Braess paradox.

In the above example, before the network update, the equilibrium travel time of route $A-C-B$ or $A-D-B$ is equal to 65 minutes, and after the update, the equilibrium travel time of route $A-C-D-B$ is equal to 80 minutes, then $(80-65)/65 < 1/3$. If before the network update, the equilibrium travel time along the route $A-C-B$ or $A-D-B$ is 60 minutes (a subroute $C-B$ trip takes 40 minutes, as well as a subroute $A-D$ trip), and after the update, the equilibrium travel time along the route $A-C-D-B$ is 80 minutes, then $(80-60)/60 = 20/60 = 1/3$.

A traffic pattern is the choice of a path by each driver, and the social cost of a given traffic pattern is the total travel time of all drivers required for using this pattern. Let there be 4 players, travel time by $C-D$ is 0, by $C-B$ or $A-D$ is 5, by $A-C$ or $D-B$ is x. Then there is scheme 1, where all players use the route $A-C-D-B$ (each player needs to travel 8 (minutes)), and there is scheme 2, where 2 players use the route $A-C-B$ (each player needs to travel $x+5=2+5=7$ (minutes)) and 2 players use the route $A-D-B$ (each player needs for the trip 7 (minutes)). The social cost of scheme 1 (single Nash equilibrium) is $4 \times 8 = 32$, and the social cost of scheme 2 (socially optimal scheme) is $4 \times 7 = 28$.

4 Conclusion

Any common network update should correspond to a certain measurable criterion of social gain, which each person understands in his or her own way. The reconfiguration of any network involves a certain coordination of all its participants.

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Data Parallelization for Solving Bimatrix Games

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Abstract

Contemporary decision-making problems are very complex and require the processing of a very large volume of data. Thus, for the mathematical modelling of these processes, it is necessary to take into account the big data problems. The data is too big to be stored and processed by a single machine. In many large-scale solutions, data is divided into partitions that can be managed and accessed separately. In order to solve such problems in real time, parallel algorithms are built and then implemented on various types of parallel computing systems. In this paper, we will analyze the ways to build parallel algorithms, and especially, data parallelization, for a class of noncooperative games, bimatrix games.

Keywords: bimatrix games, equilibrium profiles, HPC systems, parallel algorithms.

1 Matrix games and algorithms for determining equilibrium profiles in pure strategies.

We consider the bimatrix game in the following strategic form $\Gamma = \langle I, J, A, B \rangle$, where $I = \{1, 2, .., n\}$ is the line index set (the set of strategies of the player 1), $J = \{1, 2, .., m\}$ is the column index set (the set of strategies of the player 2), and $A = ||a_{ij}||_{i\in I}^{j\in J}$, $B = ||b_{ij}||_{i\in I}^{j\in J}$ are the payoff matrices of player 1 and player 2, respectively. All players know exactly the payoff matrices and the sets of strategies. So, the game is incomplete and has imperfect information. Players intend to maximize their payoffs. The matrices A and B are called *global matrices.* We denote by $NE[\Gamma]$ the set of all equilibrium profiles in the game Γ.

We will build the following point-to-set application $Br_1 : J \to 2^I$ such that, for any fixed column $j \in J$, $Br_1(j) = Arg \max_{i \in I} a_{ij} = \left\{ i^* \in I : i^* = \arg\max_{i \in I} a_{ij} \right\}$ is determined. Similarly, $Br_2 : I \to 2^J$ such that, for any fixed row $i \in I$, $Br_2(i) = Arg \max_{j \in J} b_{ij} = \left\{ j^* \in J : j^* = \arg\max_{j \in J} b_{ij} \right\}$ is determined. The graphs of Br_1 and Br_2 are denoted by $GrBr_1$ and $GrBr_2$. Then $(i^*,\ j^*) \in NE[\Gamma] \Leftrightarrow (i^*,\ j^*) \in GrBr_1 \bigcap GrBr_2$.

Using the definition, the following *basic sequential algorithm* can be used to determine the Nash equilibrium profiles in pure strategies:

1) for any fixed column $j \in J$, the sets $Br_1(j) = Arg \max_{i \in I} a_{ij}$ and $Br_2(i) = Arg \max_{j \in J} b_{ij}$ are determined;

2) the graphs $GrBr_1$, $GrBr_1$ of the point-to-set application Br_1 and Br_2 are built;

3) construct the set $NE = GrBr_1 \cap GrBr_2$.

2 Generation of sets of bimatrix subgames as a result of the division into blocks of global matrices.

We denote by R the set of processes (processors, computing elements) of a parallel computing system and by I_r and J_r – the lines (columns) of the matrices A_r and B_r, respectively, distributed to the process $r \in R$. For the characterization of the algorithms of division and distribution on matrix-like systems, we will introduce the following functions. The functions $\alpha : I \times R \to I_r \beta : J \times R \to J_r$ will be called Division and Distribution of the Elements of a Matrix (*D&DEM* function), if they verify the following property: for any $i \in I$, there is a single process r and $i_r \in I_r$, such that $\alpha(r, i) = i_r$, and similarly, for any $j \in J$, there is only one process r and $j_r \in J_r$, so that $\beta(r,j) = j_r$. Also the functions $\varphi : I_r \to I$ and $\psi : J_r \to J$ will be called Restoring the Elements of a Matrix (*REM* function), if they verify the following property: for any $r \in R$ and $i_r \in I_r$, there is a single element $i \in I$ such that $\varphi(i_r) = i$ and similarly, for any $r \in R$ and $j_r \in J_r$, there is a single element $j \in J$, so that $\psi(j_r) = j$. Thus, using the D&DEM functions, we will obtain a series of sub-matrices $\left\{A_r = \left\|a_{\alpha(r,i)\beta(r,j)}\right\|_{i\in I}^{j\in J} \equiv ||a^r_{i_r j_r}||_{i_r\in I_r}^{j_r\in J_r}\right\}_{r\in R}$, and $\left\{B_r = \left\|b_{\alpha(r,i)\beta(r,j)}\right\|_{r\in R} \equiv ||b^r_{i_r j_r}||_{i\in I_r}^{j\in J_r}\right\}_{r\in R}$ which are a result of the division and distribution of matrices A and correspondingly B. Using the REM and submatrices $\{A_r\}_{r\in R}$, $\{B_r\}_{r\in R}$, we will reconstruct the global matrices A and B. If the submatrices have the same size, then these submatrices can generate a series of sub-games $\Gamma_r = \langle I_r, J_r, A_r, B_r\rangle$ and denote by $NE[\Gamma_r]$ the set of Nash equilibrium profiles. Thus, an algorithm for dividing matrices into sub-matrices is uniquely characterized by *D&DEM* and *REM* functions.

For example, if we want to divide the matrices into consecutive n_r lines of the matrix A and m_r columns of the matrix B, the *D&DEM* and *REM* functions will show the following form: D&DEM functions

$$\alpha(r,i) = \begin{cases} i - \sum\limits_{k=0}^{r-1} n_k & \text{for } A, \\ i & \text{for } B, \end{cases}, \ \beta(r,j) = \begin{cases} j & \text{for } A, \\ j - \sum\limits_{k=0}^{r-1} m_k & \text{for } B, \end{cases}$$

REM functions

$$\varphi(r,i_r) = \begin{cases} i_r + \sum\limits_{k=0}^{r-1} n_k & \text{for } A, \\ i_r & \text{for } B, \end{cases}, \ \psi(r,j_r) = \begin{cases} j & \text{for } A, \\ j_r + \sum\limits_{k=0}^{r-1} m_k & \text{for } B. \end{cases}$$

3 Basic Parallel Algorithm to find Nash equilibrium profiles

We will present the following parallel algorithm for determining the solutions of matrix games in pure strategies.

A) Data paralelizations: using the *D&DEM* functions, each process $r \in R$ "gets" the pair of matrices A_r, B_r.

B) Each process $r \in R$, independently using the *Basic sequential algorithm*, will construct the following point-to-set applications $Br_1^r(j_r)$ for all $j_r \in J_r$ and $Br_2^r(i_r)$ for all $i_r \in I_r$, and also the set of index pairs $GrBr_1^r \cap GrBr_2^r$ and send these sets to the root process. If, as a result of the division of matrices A and B, subgames $\Gamma_r = \langle I_r, J_r, A_r, B_r \rangle$ are generated, then $NE[\Gamma_r] = GrBr_1^r \cap GrBr_2^r$;

C) Root process, using the sets $\{GrBr_1^r\}_{r \in R}$, $\{GrBr_2^r\}_{r \in R}$ and the *REM* functions, the graph $GrBr_1$ of the point-to-set application Br_1 and also the graph $GrBr_2$ of the point-to-set application Br_2 are built. The equilibrium profiles are all the profiles belonging to the intersection of the two given graphs $NE = GrBr_1 \cap GrBr_2$.

4 Conclusion

This article presents a new way to solve bimatrix games with very large matrices using parallel algorithms. The implementation of parallel algorithms depends a lot on the methods of dividing matrices into sub-matrices and their distribution on an HPC-type computing system, i.e., on achieving data parallelization. For the row and column subarrays, the described algorithms were implemented in software on the USM HPC system.

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Some Analytical Properties of the Laplace Transform of the Characteristic Function of the Three-Dimensional Markov Random Flight

Alexander D. Kolesnik

Abstract

The analytical properties of the Laplace transform of the characteristic function of the three-dimensional symmetric Markov random flight, are studied. The only singular point (ordinary pole) and the residue at this point are evaluated, which gives the first coefficient (by the negative power of complex variable) in the Laplace decomposition of the Laplace-Fourier transform of the transition density of the process.

Keywords: Markov random flight, Laplace-Fourier transform, characteristic function, singular point, ordinary pole, residue, Laurent series

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t)),\ t > 0$, be the three-dimensional symmetric Markov random flight represented by the stochastic motion of a particle that moves with constant speed c in the Euclidean space $\mathbb{R}^3$ and changes its direction at λ-Poisson distributed random time instants, choosing the initial and each new direction at random according to some probability distribution on the surface of the unit sphere.

Consider the characteristic function $H(\boldsymbol{\alpha}, t)$ of the process $\mathbf{X}(t)$ defined by $H(\boldsymbol{\alpha}, t) = \mathbb{E}\left[e^{\langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle}\right], \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3, \quad t > 0$, where $\mathbb{E}$ means the expectation and $\langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle$ is the inner product of three-dimensional vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

It is known (see [1, formula (4.6.4)], [2, page 1054]), [4, formula(45)], [5, formulas (1.6) and (5.8) for $c = 1$] that the Laplace transform of characteristic function $H(\boldsymbol{\alpha}, t)$ with respect to time variable t is given by the formula:

$$f(s) = \mathcal{L}_t \left[H(\boldsymbol{\alpha}, t) \right](s) = \frac{\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}, \tag{1}$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}, \quad \operatorname{Re} s > 0.$

The main goal of this article is to briefly outline some analytical properties of function (1). The results presented here are a part of a more general research related to the asymptotic behaviour of the characteristic function $H(\boldsymbol{\alpha}, t)$. The following lemma yields the singular point of this function, which is the ordinary pole of first order.

Lemma 1. *Function (1) has the single singular point (ordinary pole)*

$$s^* = -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right). \tag{2}$$

Proof. To find the singular point of function (1), one needs to solve with respect to s the equation

$$\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right) = \frac{c\|\boldsymbol{\alpha}\|}{\lambda}. \tag{3}$$

In the inverse tangent function on the left-hand side of equation (3), we take its principal branch. Under such a choice, this function is holomorphic (analytical) and single-valued and, therefore, equation (3) has the unique solution. We also see that the right-hand side of (3) is a real-valued positive number. Hence, one can conclude that the inverse tangent function on the left-hand side of (3) must also take real values and, therefore, the variable s solving this equation must be a real number (that is, its imaginary part is zero). Then solving equation (3) with real values of variable s, we easily obtain the unique solution given by (2). □

Lemma 1 enables us to evaluate the residue of function (1) at the pole s^* given by (2).

Theorem 1. *The residue of function (1) at the ordinary pole s^* is given by the formula:*

$$\operatorname{res} f(s^*) = \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2 \left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda} \right). \tag{4}$$

Proof. Since s^* is the ordinary pole, then residue (4) can be evaluated by means of the standard relation (see [3, page 79, formula (4)]):

$$\begin{aligned}
\operatorname{res} f(s^*) &= \lim_{s \to s^*} \left[(s - s^*)\, f(s) \right] \\
&= \lim_{s \to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\left(s + \lambda - c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda} \right) \right) \right. \\
&\qquad \left. \times \frac{\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)} \right] \\
&= \lim_{s \to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s + \lambda} \right) \right] \\
&\qquad \times \lim_{s \to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\frac{s + \lambda - c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)}{c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)} \right].
\end{aligned}$$

The first limit is obviously equal to $c\|\boldsymbol{\alpha}\|/\lambda$. By applying L'Hôpital's rule to the second limit, we get:

$$\begin{aligned}
\operatorname{res} f(s^*) &= \frac{c\|\boldsymbol{\alpha}\|}{\lambda} \lim_{s \to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[\frac{(s + \lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}{\lambda\, c\|\boldsymbol{\alpha}\|} \right] \\
&= \frac{1}{\lambda^2} \lim_{s \to -\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right)} \left[(s + \lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2 \right] \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \left[\operatorname{ctg}^2 \left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda} \right) + 1 \right] \\
&= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2 \left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda} \right),
\end{aligned}$$

proving (4).

This formula can also be obtained by other way. Taking into account that

$$\left(c\|\boldsymbol{\alpha}\| - \lambda \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{s+\lambda}\right)\right)'_s = \frac{\lambda\, c\|\boldsymbol{\alpha}\|}{(s+\lambda)^2 + (c\|\boldsymbol{\alpha}\|)^2}$$

and applying [3, page 79, formula (5)], we get:

$$\begin{aligned}\operatorname{res} f(s^*) &= \operatorname{arctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{-\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + \lambda}\right) \\ &\quad\times \left[\frac{\lambda\, c\|\boldsymbol{\alpha}\|}{\left(-\lambda + c\|\boldsymbol{\alpha}\| \operatorname{ctg}\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + \lambda\right)^2 + (c\|\boldsymbol{\alpha}\|)^2}\right]^{-1} \\ &= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2}\left[\operatorname{ctg}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) + 1\right] \\ &= \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right),\end{aligned}$$

and we again arrive at (4). The theorem is proved. □

Corollary 1. Theorem 1 yields the first coefficient

$$\xi_{-1} = \frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2} \operatorname{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right) \tag{5}$$

in the Laurent decomposition $f(s) = \sum_{k=-1}^{\infty} \xi_k\, (s-s^*)^k$ of function $f(s)$ given by (1), which is the Laplace transform of the characteristic function of the three-dimensional symmetric Markov random flight $\mathbf{X}(t)$.

Corollary 2. Let D be an arbitrary domain in complex plane $\mathbb{C}$ such that the pole s^* is the interior point of D, and let C be the boundary of D. Since s^* is the only singular point of function $f(s)$ and it lies strictly inside D, then $f(s)$ is holomorphic (analytical) in all the interior points of D, except s^*, and continuous on its boundary C. Then, according to the residues theorem,

$$\int_C f(s)\,ds = 2\pi i\,\frac{(c\|\boldsymbol{\alpha}\|)^2}{\lambda^2}\,\mathrm{cosec}^2\left(\frac{c\|\boldsymbol{\alpha}\|}{\lambda}\right),$$

where the boundary C is circuiting in positive direction (that is, counterclock-wise).

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Characteristic Function of the Markov Random Flight in Higher Dimensions

Alexander D. Kolesnik

Abstract

Two series representations of the characteristic function of the multidimensional symmetric Markov random flight, are presented. These series are the decompositions of the characteristic function with respect to Bessel functions and with respect to time variable, whose coefficients are given by recurrent relations, as well as in the form of special determinants. Basing on these series representations, an asymptotic formula for the second moment function of the process, is obtained.

Keywords: Markov random flight, characteristic function, series representation, recurrent relations, moment function, asymptotic formula

Let $\mathbf{X}(t) = (X_1(t), \dots, X_m(t)),\ t > 0$, be the m-dimensional symmetric Markov random flight represented by the stochastic motion of a particle that moves with constant speed c in the Euclidean space $\mathbb{R}^m,\ m \geq 3$, and changes its direction at λ-Poisson distributed random time instants, choosing the initial and each new direction at random according to some probability distribution on the surface of the unit $(m-1)$-dimensional sphere.

Consider the characteristic function $H(\boldsymbol{\alpha}, t)$ of the process $\mathbf{X}(t)$ defined by

$$H(\boldsymbol{\alpha}, t) = \mathbb{E}\left[e^{\langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle}\right], \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m, \quad t > 0,$$

where $\mathbb{E}$ means the expectation and $\langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle$ is the inner product of m-dimensional vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

Theorem 1. *For arbitrary dimension $m \geq 3$, the characteristic function $H(\boldsymbol{\alpha}, t)$ of the m-dimensional symmetric Markov random flight $\mathbf{X}(t)$ has the following series representation:*

$$H(\boldsymbol{\alpha}, t) = e^{-\lambda t} \sqrt{\pi} \sum_{n=1}^{\infty} \frac{\zeta_n(\boldsymbol{\alpha})}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{t}{2c\|\boldsymbol{\alpha}\|}\right)^{(n-1)/2} J_{(n-1)/2}(ct\|\boldsymbol{\alpha}\|), \quad (1)$$

where $J_\nu(z)$ are Bessel functions and the coefficients $\zeta_n = \zeta_n(\boldsymbol{\alpha})$ are given by the recurrent relation:

$$\zeta_1 = 1, \qquad \zeta_n = \xi_n + \lambda \sum_{k=1}^{n-1} \zeta_{n-k}\, \xi_k, \qquad n \geq 2, \quad (2)$$

with

$$\xi_n = \begin{cases} 0, & \text{if } n = 2r, \\ \dfrac{(2r-1)!!}{(2r)!!} \, \dfrac{m-2}{2r+m-2} \, (c\|\boldsymbol{\alpha}\|)^{2r}, & \text{if } n = 2r+1, \end{cases} \quad (3)$$

for $r = 0, 1, 2, \ldots,$ where $(2r)!! = 2 \cdot 4 \cdot 6 \cdot \ \ldots \ \cdot (2r)$, $(2r-1)!! = 1 \cdot 3 \cdot 5 \cdot \ \ldots \ \cdot (2r-1)$, $r \geq 0$, $0!! = (-1)!! \overset{def}{=} 1$.

One can, instead of recurrent relation (2), give an explicit formula for evaluating the coefficients ζ_n, $n \geq 2$, in the form of the following $(n \times n)$-determinant:

$$\zeta_n = (-1)^{n+1} \times \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \ldots 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 \ldots 0 & 0 \\ \xi_3 & 0 & -\lambda & 1 & 0 & 0 \ldots 0 & 0 \\ 0 & -\lambda\xi_3 & 0 & -\lambda & 1 & 0 \ldots 0 & 0 \\ \xi_5 & 0 & -\lambda\xi_3 & 0 & -\lambda & 1 \ldots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\ \xi_{n-1} & -\lambda\xi_{n-2} & -\lambda\xi_{n-3} & -\lambda\xi_{n-4} & -\lambda\xi_{n-5} & 0 \ldots -\lambda & 1 \\ \xi_n & -\lambda\xi_{n-1} & -\lambda\xi_{n-2} & -\lambda\xi_{n-3} & -\lambda\xi_{n-4} & -\lambda\xi_{n-5} \ldots 0 & -\lambda \end{vmatrix} \quad (4)$$

for arbitrary $n \geq 2$.

One can also obtain a series representation of $H(\boldsymbol{\alpha}, t)$ with respect to the powers of time variable t. This result is given by the following theorem.

Theorem 2. *For arbitrary dimension $m \geq 3$, the characteristic function $H(\boldsymbol{\alpha}, t)$ of the m-dimensional symmetric Markov random flight $\mathbf{X}(t)$ has the following series representation:*

$$H(\boldsymbol{\alpha}, t) = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(\boldsymbol{\alpha})}{n!} \, t^n, \tag{5}$$

where the coefficients $\gamma_n = \gamma_n(\boldsymbol{\alpha})$ are given by the recurrent relation:

$$\gamma_1 = 1, \qquad \gamma_n = \theta_n + \lambda \sum_{k=1}^{n-1} \gamma_{n-k} \, \theta_k, \qquad n \geq 2, \tag{6}$$

and

$$\theta_n = \begin{cases} 0, & \text{if } n = 2r, \\ (-1)^r \, \dfrac{\left(\frac{1}{2}\right)_r}{\left(\frac{m}{2}\right)_r} \, (c\|\boldsymbol{\alpha}\|)^{2r}, & \text{if } n = 2r+1, \end{cases} \qquad r = 0, 1, 2, \dots \tag{7}$$

where $(a)_r$ is the Pochhammer symbol.

The coefficients γ_n can also be represented in the explicit form of the following $(n \times n)$-determinant:

$$\gamma_n = (-1)^{n+1} \times \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \dots 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 \dots 0 & 0 \\ \theta_3 & 0 & -\lambda & 1 & 0 & 0 \dots 0 & 0 \\ 0 & -\lambda\theta_3 & 0 & -\lambda & 1 & 0 \dots 0 & 0 \\ \theta_5 & 0 & -\lambda\theta_3 & 0 & -\lambda & 1 \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\ \theta_{n-1} & -\lambda\theta_{n-2} & -\lambda\theta_{n-3} & -\lambda\theta_{n-4} & -\lambda\theta_{n-5} & 0 \dots -\lambda & 1 \\ \theta_n & -\lambda\theta_{n-1} & -\lambda\theta_{n-2} & -\lambda\theta_{n-3} & -\lambda\theta_{n-4} & -\lambda\theta_{n-5} \dots 0 & -\lambda \end{vmatrix} \tag{8}$$

for arbitrary $n \geq 2$.

The proofs of Theorems 1 and 2 are based on evaluating the inverse Laplace transform in the explicit formula for the Laplace-Fourier transform of the distribution of $\mathbf{X}(t)$ obtained in [1, 2].

The above results can be applied to evaluate the moment function by means of well-known formula connecting the characteristic function and the moment function of a stochastic process. In particular, for the first and second moment functions of the three-dimensional symmetric Markov random flight $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t))$, the following theorem holds true.

Theorem 3. *For arbitrary $t > 0$, the first mixed moment of the three-dimensional symmetric Markov random flight $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t))$ is identically equal to zero, that is,*

$$\mu_{\mathbf{q}_1}(t) = \mu_{(1,1,1)}(t) \equiv 0. \tag{9}$$

For the second mixed moment function of $\mathbf{X}(t)$, the following asymptotic relation holds:

$$\mu_{\mathbf{q}_2}(t) = \mu_{(2,2,2)}(t) = e^{-\lambda t}\left[\frac{1}{105}\,(ct)^6 + \lambda\,\frac{44}{11025}\,c^6 t^7 + o(t^7)\right]. \tag{10}$$

Asymptotic formula (10) provides a very good approximation of the second mixed moment, especially for small values of time t. For example, for the time value $t = 0.3$, relation (10) yields (for $c = 3$, $\lambda = 5$) the estimate: $\mu_{(2,2,2)}(0.3) \approx 0.00183921$ and the error of this estimate does not exceed the value 0.00001464 multiplied by some constant.

For any multi-index of the form $(0,\dots,0,2,0,\dots,0)$ of arbitrary dimension $m \geq 3$, the respective moment function is given by the explicit formula:

$$\mu_{(0,\dots,0,2,0,\dots,0)}(t) = \frac{2}{m}\,\frac{c^2}{\lambda^2}\left(e^{-\lambda t} + \lambda t - 1\right), \qquad t > 0, \quad m \geq 3. \tag{11}$$

We see that (11) is a nonlinear monotonously increasing function which acquires almost linear growth, as time increases.

Under the standard Kac scaling condition

$$c \to \infty, \qquad \lambda \to \infty, \qquad \frac{c^2}{\lambda} \to \rho, \quad \rho > 0, \tag{12}$$

function (11) turns into:

$$\lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \mu_{(0,\dots,0,2,0,\dots,0)}(t) = \frac{2}{m} \lim_{\substack{c,\,\lambda\to\infty \\ (c^2/\lambda)\to\rho}} \left[\frac{c^2}{\lambda} \left(\frac{e^{-\lambda t}}{\lambda} + t - \frac{1}{\lambda} \right) \right] = \frac{2\rho}{m}\, t,$$

and this is the variance of the m-dimensional homogeneous Brownian motion with zero drift and diffusion coefficient $\sigma^2 = 2\rho/m$. This entirely accords with already known result concerning the limiting behaviour of the m-dimensional symmetric Markov random flight under the standard Kac scaling condition (12) (see [1, Theorem 4.8.1] and [2, Theorem 4]).

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Homogeneous Linear Recurrent Games

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Abstract

Games defined on deterministic systems with homogeneous linear recurrent dynamic are studied. It is proved that they represent homogeneous linear recurrent systems and an efficient method for determining the generating vector is proposed. This enables the usage of the advanced properties of homogeneous linear recurrences for their algebraic characterization.

Keywords: dynamical process, homogeneous linear recurrence, game.

1 Introduction and Problem Formulation

Let L be a discrete system with the set of states $V \subseteq \mathbb{C}$. At every discrete moment of time $t \in \mathbb{N}$, the state of the system is $v(t) \in V$. The state $v(t)$ of the system for every t, $0 \leq t < m$, is known and, at every moment of time $t \geq m$, the dynamic of the system satisfies the relation $v(t) = \sum_{k=0}^{m-1} q_k v(t-1-k)$, where $q_k \in K$, $k = \overline{0, m-1}$, and $K \subseteq \mathbb{C}$. The system L represents a discrete system with homogeneous linear recurrent dynamic over K, generated by generating vector $q = (q_k)_{k=0}^{m-1}$ and initial state $I_m^{[v]} = (v(t))_{t=0}^{m-1}$. These homogeneous linear recurrent dynamics were deeply investigated in [2] and their advanced algebraic properties were studied in [1].

Next, in this paper, a variation is studied. The following game is considered. Initially, each player $\mathcal{P}_\ell$ defines his stationary strategy, represented by one generating vector $q^{(\ell)} \in K^{m_\ell}$ and one initial state $I_{m_\ell}^{[v]}$, $\ell = \overline{0, r-1}$.

The game is started by first player $\mathcal{P}_0$. At every moment of time t, the system passes to the state $v(t)$ accordingly to the strategy of the current player $\mathcal{P}_{t\ mod\ r}$. After the last player $\mathcal{P}_{r-1}$, the first player $\mathcal{P}_0$ acts on the system evolution and the game continues in this way.

Our goal is to prove that the defined game represents a homogeneous linear recurrent system and to provide an efficient approach for determining its generating vector, such that the previously developed methods for its algebraic characterization to be enabled for usage.

2 Homogeneous Linear Recurrences

As mentioned above, the sequence $a = \{a_n\}_{n=0}^{\infty}$ represents a homogeneous linear m-recurrence on the set K if $\exists q = (q_k)_{k=0}^{m-1} \in K^m$ such that $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$, $\forall n \geq m$, where q is the generating vector and $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ is the initial state of the sequence a. It is denoted by $Rol^*[K][m]$ the set of all homogeneous linear m-recurrences on the set K. The set $G^*[K][m](a)$ represents the set of generating vectors of length m of the sequence $a \in Rol^*[K][m]$.

The function $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ is the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$ and the function $G_t^{[a]}(z) = \sum_{n=0}^{t-1} a_n z^n$ is the partial generating function of order t of the sequence a. We consider the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$. For an arbitrary non-zero α, the polynomial $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ represents a characteristic polynomial of the sequence a of order m. We denote by $H^*[K][m](a)$ the set of characteristic polynomials of order m of the sequence $a \in Rol^*[K][m]$.

The sequence a is called m-minimal on the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, for all $t < m$. The number m is called the dimension of sequence a on the set K (denoted $dim[K](a) = m$).

Next we will recall several important theorems presented in [2].

Theorem 1. *If $a \in Rol^*[\mathbb{C}][m]$ and $q \in G^*[\mathbb{C}][m](a)$, then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)}.$$

Theorem 2. *Consider that $G^{[a]}(z) = \dfrac{A(z)}{B(z)}$ is a rational fraction with $deg(A(z)) = M$, $B(z) = 1 - z \sum_{k=0}^{m-1} q_k z^k$ and $q_k \in K$, $k = \overline{0, m-1}$. Then $a \in Rol^*[K][t]$, $B(z) \in H^*[K][t](a)$, where $t = \max\{m, M+1\}$.*

Theorem 3. *If $a \in Rol^*[\mathbb{C}][m]$ is a sequence with at least one non-zero element, then $dim[\mathbb{C}](a) = R$ and $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, where $R = rank(A_m^{[a]})$, $A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}}$, $f_n^{[a]} = (a_k)_{k=\overline{n,2n-1}}$, $\forall n \geq 1$ and the vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ represents the unique solution of the system $A_R^{[a]} x^T = (f_R^{[a]})^T$.*

3 Homogeneous Linear Recurrent Games

We consider the game defined in Section 1. The dynamic generated by this game is $a = (a_n)_{n=0}^{\infty}$, where $a_n = v(n)$, $n = \overline{0, \infty}$.

The players $\mathcal{P}_0$, $\mathcal{P}_1$, $\ldots$, $\mathcal{P}_{r-1}$ act consecutively on the system evolution and the game continues in the loop after that. In this way, every player $\mathcal{P}_\ell$ directly contributes only at establishment of the states $a_{nr+\ell}$, $n = \overline{0, \infty}$, for each $\ell \in \{0, 1, \ldots, r-1\}$.

Let $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function of the sequence $a = (a_n)_{n=0}^{\infty}$. The contribution of each player $\mathcal{P}_\ell$ to this generating function is the function $g_\ell(z) = \sum_{n=0}^{\infty} a_{nr+\ell}\, z^{nr+\ell}$, $\ell = \overline{0, r-1}$. Based on these contributions, the generating function becomes $G^{[a]}(z) = \sum_{\ell=0}^{r-1} g_\ell(z)$.

The function $g_\ell(z)$ can be written in the following way:

$$g_\ell(z) = \sum_{n:\ rn+\ell<m_\ell} a_{rn+\ell}\, z^{rn+\ell} + \sum_{n:\ rn+\ell\geq m_\ell} z^{rn+\ell} \sum_{k=0}^{m_\ell-1} q_k^{(\ell)} a_{rn+\ell-k-1} =$$

$$= W^{(\ell)}_{m_\ell-1}(z) + \sum_{k=0}^{m_\ell-1} q_k^{(\ell)} z^{k+1} g_{s(\ell,k)}(z),$$

where $deg(W^{(\ell)}_{m_\ell-1}(z)) \leq m_\ell-1$, $s(\ell,k) = (\ell-k-1)\ mod\ r$, $\ell = \overline{0, r-1}$.

These relations can be rewritten as $A(z)(g(z))^T = (W(z))^T$, where $A(z) = (a_{ij}(z))_{i,j=0}^{r-1}$, $g(z) = (g_j(z))_{j=0}^{r-1}$, $W(z) = (W^{(j)}_{m_j-1}(z))_{j=0}^{r-1}$ and $deg(a_{ij}(z)) \leq m_j$, $deg(W^{(j)}_{m_j-1}(z)) \leq m_j - 1$, $i,j = \overline{0, r-1}$. Using Cramer's rule, $g_\ell(z) = \dfrac{A_\ell(z)}{B(z)}$, where $deg(A_\ell(z)) < m$, $deg(B(z)) \leq m$ and $m = \sum\limits_{j=0}^{r-1} m_j$, $\ell = \overline{0, r-1}$. Calculating $G^{[a]}(z) = \sum\limits_{\ell=0}^{r-1} g_\ell(z)$, and applying Theorem 2, we obtain that $a \in Rol^*[\mathbb{C}][m]$. The minimal generating vector can be determined using Theorem 3.

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Strongly Polynomial Primal-Dual Algorithms for Concave Cost Combinatorial Optimization Problems*

Thomas L. Magnanti, Dan Stratila

Abstract

We introduce an algorithm design technique for a class of combinatorial optimization problems with concave costs. This technique yields a strongly polynomial primal-dual algorithm for a concave cost problem whenever such an algorithm exists for the fixed-charge counterpart of the problem.

Using our technique, we obtain a 1.61-approximation algorithm for the concave cost facility location problem, an exact algorithm for the concave cost lot-sizing problem, and a 4-approximation algorithm for the joint replenishment problem with general concave individual ordering costs.

Keywords: Primal-dual algorithms, concave costs, facility location, lot-sizing, joint replenishment problems.

1 Concave Cost Facility Location

In the concave cost facility location problem, there are m customers and n facilities. Each customer i has a demand $d_i > 0$, and needs to be connected to a facility to satisfy this demand. Connecting a customer i to a facility j incurs a connection cost $c_{ij}d_i$; we assume that the c_{ij} are nonnegative and satisfy the metric inequality. Each facility j has a nondecreasing concave cost function $\phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, and we assume

*This abstract is based on [8, 6]. A complete bibliography for this abstract can be found in [8, 6].

without loss of generality that $\phi_j(0) = 0$. Let $x_{ij} = 1$ if customer i is connected to facility j, and $x_{ij} = 0$ otherwise. The cost at facility j is a function of the total demand at j, that is $\phi_j\big(\sum_{i=1}^m d_i x_{ij}\big)$, and the total cost is $\sum_{j=1}^n \phi_j\big(\sum_{i=1}^m d_i x_{ij}\big) + \sum_{i=1}^m \sum_{j=1}^n c_{ij} d_i x_{ij}$. The goal is to minimize the total cost.

Let $[n] = \{1, \dots, n\}$. The concave cost facility location problem can be written as a mathematical program:

$$\min\Big\{\sum_{j=1}^n \phi_j \left(\sum_{i=1}^m d_i x_{ij}\right) + \sum_{i=1}^m \sum_{j=1}^n c_{ij} d_i x_{ij} \;\Big|\; \sum_{j=1}^n x_{ij} = 1, x_{ij} \geq 0, i \in [m], j \in [n]\Big\}. \quad (1)$$

We omit the constraints $x_{ij} \in \{0, 1\}$, which are automatically satisfied at any vertex of the feasible polyhedron. Since the objective is concave, this problem always has a vertex optimal solution.

We interpret each concave function ϕ_j as a piecewise-linear function with an infinite number of pieces. For each $p > 0$, we introduce a tangent $f_{jp} + s_{jp}\xi_j$ to ϕ_j at p, with $s_{jp} = \phi_j'(p)$, and $f_{jp} = \phi_j(p) - p s_{jp}$. Here $\phi_j'(p)$ is the right derivative, which exists as ϕ_j is concave.

We have $\phi_j(\xi_j) = \min\{f_{jp} + s_{jp}\xi_j : p > 0\}$ for $\xi_j > 0$. Problem (1) can be written as a classical facility location problem with m customers and an infinite number of facilities. (For background on the classical facility location problem, see [8, 6].) Each tangent p to cost function ϕ_j of facility j in problem (1) corresponds to a facility $\{j, p\}$ in the resulting problem. The resulting infinitely-sized integer program is:

$$\min\Big\{ \sum_{j=1}^n \sum_{p\geq 0} f_{jp} y_{jp} + \sum_{i=1}^m \sum_{j=1}^n \sum_{p \geq 0} (c_{ij} + s_{jp}) d_i x_{ijp} \;\Big|\; \sum_{j=1}^n \sum_{p \geq 0} x_{ijp} = 1, x_{ijp} \leq y_{jp}, \; y_{jp} \in \{0, 1\}, i \in [m], j \in [n], p \geq 0\Big\}. \quad (2)$$

Consider the 1.61-approximation algorithm for classical facility location of Jain et al. [2], which we will call Algorithm FLPD. Of course, we cannot run this algorithm on problem (2) directly, as it is infinitely-sized. Instead, we obtain technical results that enable us to execute Algorithm FLPD on problem (2) implicitly. More specifically, we devise an algorithm that takes problem (1) as input, runs in strongly polynomial time, and produces the same assignment of customers to facilities as if Algorithm FLPD were run on problem (2). We call this algorithm ConcaveFLPD.

Theorem 1. *Algorithm* ConcaveFLPD *is a 1.61-approximation algorithm for the concave cost facility location problem, with a running time of* $O(m^3 n + mn \log n)$.

Using piecewise-linear approximations, we also obtained a $1.4991+\epsilon$ approximation algorithm for concave cost facility location [4, 8, 5]. The running time is polynomial in $1/\epsilon$ and the size of the input, but is not strongly polynomial. Independently, Romeijn et al. [7] developed strongly polynomial 1.61 and 1.52-approximation algorithms for concave cost facility location, each with a running time of $O(n^4 \log n)$, where n is the higher of the number of customers and facilities.

2 Concave Cost Inventory Problems

We obtain a strongly polynomial exact algorithm for the concave cost lot-sizing problem with a running time of $O(n^2)$, where n is the number of time periods. To develop this algorithm, we use our technique together with the algorithm of Levi et al. [3] for the classical lot-sizing problem. While the running time matches that of the fastest previous algorithm [1], our main goal is to use this algorithm as a stepping stone in the development of our approximation algorithm for the joint replenishment problem (JRP) with concave individual ordering costs.

We obtain a strongly polynomial 4-approximation algorithm for the JRP with concave individual ordering costs. We call this algorithm ConcaveJRPPD. To develop it, we use our technique together with the 2-approximation algorithm of Levi et al. [3] for the classical JRP.

Theorem 2. *Algorithm* ConcaveJRPPD *is a 4-approximation algorithm for the JRP with concave individual ordering costs.*

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Informational Extended Games And Their Applications

Ludmila Novac

Abstract

In this article, we analyse informational extended games, i.e., games in which the players choose their actions simultaneously, with assumption that they have some information about the future strategies which will be chosen by other players. For all informational extended games of this type we assume that players' payoff functions are common knowledge. Under these assumptions we define the noncooperative informational extended games and analyse Nash equilibrium. As a particular case of the non-cooperative informational extended games, we analyse the class of bimatrix informational extended games and we present an example of game in order to show the possibility to use the information for this class of games.

Keywords: Informational extended games, Nash equilibrium, best response mapping, point-to-set mapping, fixed points.

1 Introduction

Using the general concept of the Informational Extended Games (IEG) for the non-cooperative games and using some models of informational extended games (as a particular case of Howard meta-games, see Howard [1]), we extend the study of noncooperative informational extended games. The informational extension concept for games has on its basis the assumptions that the participants of the game have possibility to send and to receive (or to guess) some information about the chosen strategies of other participants and about their behavior.

We construct a type of informational extended game with n players, for which we give the explicit definitions. The main problem which was formulated is to define the conditions of the Nash equilibrium existence for this type of informational extended game.

2 Strategic Form Games and Nash Equilibria

In this section, we analyse games in which the players choose their actions simultaneously (without knowledge of other player choices). The game assume that players' payoff functions are common knowledge.

Definition 1. *A strategic form of the game consists of: a finite set of players $I = \{1, 2, ..., n\}$, players' action spaces (the sets of strategies), denoted by X_i, $i \in I$; and players' payoff functions $H_i : X \to R, i \in I$, where $X = X_1 \times ... \times X_n$. We refer to such a game as the tuple $\langle I, (X_i)_{i\in I}, (H_i)_{i\in I}\rangle$ denoted by Γ.*

An outcome is an action profile $(x_1, x_2, ..., x_n)$, and the outcome space is $X = \prod_{i\in I} X_i$. The game is common knowledge among the players. One of the most common interpretations of Nash equilibrium (introduced by John Nash in 1950) is that it is a steady state in the sense that no rational player has an incentive to unilaterally deviate from it. Let us consider $x_{-i} \equiv (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and $(x_{-i}, y_i) \equiv (x_1, x_2, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)$.

Definition 2. [3] *A Nash equilibrium of the game Γ is an action profile $x^* \in X$ such that, for every $i \in I$:*

$$H_i(x^*) \geq H_i(x^*_{-i}, x_i) \forall x_i \in X_i.$$

In the following, by $F : X \rightrightarrows 2^Y$ we will denote a point-to-set mapping, were 2^Y is the set of all subsets of Y. A fixed point of the point-to-set mapping $F : X \rightrightarrows 2^Y$ is a point $x^* \in X$ such that $x^* \in F(x^*)$.

The graph for the application F is the set: $gr(F) = \{(x, y) \in X \times Y | x \in X, y \in F(x)\}$. This set can contain some points or can be an empty set. Another and sometimes more convenient way of defining Nash equilibrium is via the best response correspondences $Br_i : \prod_{j \in I \setminus i} X_j \rightrightarrows X_i$ such that

$$Br_i(x_{-i}) = \{x_i \in X_i : H_i(x) \geq H_i(x_{-i}, x_i') for \forall x_i' \in X_i\}. \qquad (1)$$

Definition 3. *A Nash equilibrium is an action profile x^* such that $x_i^* \in Br_i(x_{-i}^*)$ for all $i \in I$.*

If the sets X_i are compacts and the functions H_i are continuous, then the best response set (1) for the player i can be represented by: $Br_i(x_{-i}) =$

$$= Arg \max_{x_i \in X_i} H_i(x_{-i}, x_i) = \{x_i \in X_i | H_i(x_{-i}, x_i) = \max_{y_i \in X_i} H_i(x_{-i}, y_i)\}.$$

Given a strategic form of the game $\Gamma \equiv \langle I, (X_i) i \in I, (H_i) i \in I \rangle$, the set of Nash equilibria is denoted by $NE(\Gamma)$.

Using the best response sets of the players, we consider the point-to-set mapping $Br : \prod_{i \in I} X_i \rightrightarrows 2^X$ such that $Br = (Br_1, Br_2, ..., Br_n)$.

Then we can easily prove that $x^* \in NE(\Gamma) \leftrightarrows x^*$ is a fixed point of the set-valued mapping Br, i.e. $x^* \in Br(x^*)$.

3 Nash equilibria in the noncooperative informational extended games with n players

We analyse a static game with n players:

$$\Gamma = \langle I, X_i, (i = 1, ..., n), H_i, (i = 1, ..., n) \rangle, \qquad (2)$$

where $I = \{1, 2, ..., n\}$ is the set of players, the set of strategies for the i-th player is denoted by $X_i, (i = 1, ..., n)$, and the payoff functions are defined by: $H_i : \prod_{i \in I} X_i \to R, (i = 1, ..., n)$.

Next we analyse a static informational extended game with n players. In this informational extended game, we consider that each player is informed of the strategies of the other players which will be chosen. In this case, the sets of the strategies for each player are sets of functions defined on the product of the sets of strategies of the rest players from the initial game (2). The game is realized as follows: the strategies are chosen simultaneously by players (with assumption that each of them knows which strategies will be chosen by all other players), after that each of players determines his payoff and the game is over. This informational extended game can be described in the normal form by: ${}_n\Gamma = \langle I, \overline{X}_i, i = \overline{1,n}, \overline{H}_i, i = \overline{1,n}\rangle$, where the sets of strategies for players are defined by: $\overline{X}_i = \{\phi_i : \prod_{j\in I, j\neq i} X_j \to X_i\}, i = \overline{1,n}$, where $\prod_{j\in I, j\neq i}, X_j = X_1 \times ... \times X_{i-1} \times X_{i+1} \times ... \times X_n$. The payoff functions are defined on the product of the extended sets of strategies: $\overline{H}_i : \prod_{i\in I} \overline{X}_i \to R, (i = \overline{1,n})$.

For this informational extended game, we have the conditions of Nash equilibrium (see Novac, [5]). We apply the fixed point theorem to prove the following theorem of the Nash equilibrium existence for the informational extended game ${}_n\Gamma$ with n players.

Theorem 1. *[5] Let us consider that for the game ${}_n\Gamma$ the next conditions hold: 1) the sets $X_i \neq \emptyset, (i = \overline{1,n})$ are compacta of Banach spaces, 2) the sets of functions $\overline{X}_i \subset C(\prod_{j\in I, j\neq i} X_j, X_i), i = \overline{1,n}$ are uniformly bounded and the functions from the sets $\overline{X}_i$ are equicontinuous; 3) the payoff functions $H_i(\cdot), i = \overline{1,n}$ are continuous on the compactum $\prod_{i\in I} X_i$ and the functions $\overline{H}_i(\cdot), i = \overline{1,n}$ are concave on $\overline{X}_i$ for $\forall\phi_{-i}$, respectively. Then $NE({}_n\Gamma) \neq \emptyset$.*

4 Applications of the bimatrix informational extended games – Entering a new market

One of the most common interpretations of the "informational extended set of strategies" can be described by the next example. Let us consider an informational extended game with two players. A trivial example is the decision about entering a new market. Suppose the company, which acts as a monopoly in any market, and another outsider company which have to decide on joining or not joining the market. If in a static game, the outsider company would have some information about the reaction of the company, which acts as a monopolist, then it could chose a better strategy in order to avoid some unexpected situation and not to make his partner to act aggressively. Next we give the mathematical model of the game. Let us consider the bi-matrix game in the normal form: $\Gamma = \langle N, X, Y, A, B \rangle$, where N is the set of players: $N = \{1 = outsidercompany, 2 = monopolycompany\}$, $X = \{1, 2, ..., m\}$ and $Y = \{1, 2, ..., n\}$ – are the sets of strategies for players, A and B are the payoff matrices for players: $A = \{a_{ij}\}, B = \{b_{ij}\}, (i \in X, j \in Y)$. Suppose the sets of strategies are:

- for the Outsider company: $X = \{$ *1= enter the market in partnership, 2= enter the market in competition* $\}$,
- for the Monopoly company: $Y = \{$ *1= acceptance fair competition (loyalty), 2= acceptance partnership, 3= not accept competition, 4= not accept partnership* $\}$.

We can give a dynamical representation of this game. The tree representation of the game in this case can be given as in the picture below (Fig.1).

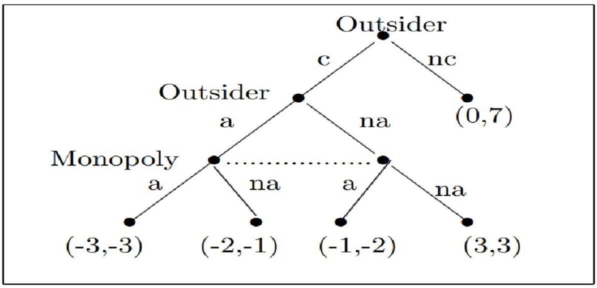

Figure 1. The game "Entering a new market"

If the Outsider company will know previously the strategy that will be applied by the Monopoly company, then he can chose the correct strategy in order to avoid some unexpected situation. So we can define this situation as a bi-matrix informational extended game.

5 Conclusion

The results explained above show that if the amount of the information increases for all participants, then the Nash equilibria sets are larger.

This treatment confirms the importance of the information' possession in all circumstances in the case of the make-decision problems and assure the best result. For the games with two players and discreet sets of strategies, we can define the bi-matrix informational extended games (see Kukushkin and Morozov, [2]) for which it is proved that the set of Nash equilibria is nonempty (see Novac, [4]). The theorem about the conditions of single Nash equilibrium existence for the bi-matrix informational extended games was stated and proved in (Novac, [4]). Our aim is to indicate the importance of the information' possession for all make-decision problems.

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PERSIST: A New Probabilistic Model For Data Denoising

Sergiu Scrob

Abstract

The paper proposes a more efficient solution for reducing the data noise level, using probability estimation for radius-based spatial inference and sampling techniques.

Keywords: data denoising, probabilistic model, radius-based spatial inference, spatial probability.

1 Introduction

The majority of clustering and classification models from scientific library like **Scikit-learn**, fails to satisfactorily denoise the data shown in Figure 1. The dataset consists of 50 subsets of 100k points positioned inside and outside the figure, forming two classes of points.

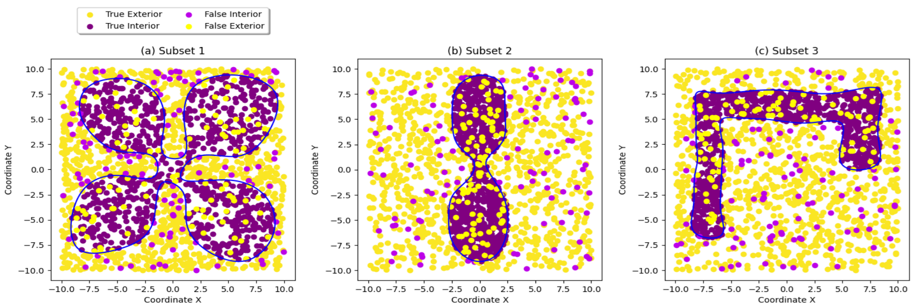

Figure 1. Examples of the first three subsets from the dataset

2 Defining a new denoising model

The new probabilistic model named PERSIST: Probability Estimation for Radius-based Spatial Inference and Sampling Techniques, aims to reduce the data noise level by using the concepts of spatial and frequency probabilities, which represents the proximity and density of the points in the vicinity of the centroid.

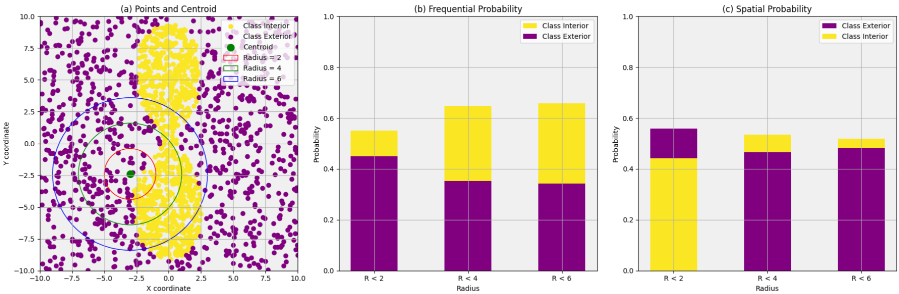

Figure 2. Local Probability Distributions by Centroid's Radius

2.1 The mathematical model

Let: $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$,$n \in \mathbb{N}$; $\mathbf{P_I} = \{(x_i, y_i) \mid (x_i, y_i) \in P, label_i = 1\}$; $\mathbf{P_E} = \{(x_i, y_i) \mid (x_i, y_i) \in P, label_i = 2\}$; $\mathbf{r}$ is the radius from centroid; $c = \{c_x, c_y\}$ as centroid point, $p = \{p_x, p_y\}$ as any point, where $c, p \in P$. The $proximity(c, P, r)$ as $\mathbb{R}^2 \times \mathcal{P}(\mathbb{R}^2) \times \mathbb{R} \to \mathcal{P}(\mathbb{R}^2) : \mathcal{N} \leftarrow \mathcal{N} \cup \mathcal{N}'$ where $\mathcal{P}(\mathbb{R}^2)$ is the power set of $\mathbb{R}^2$ and $\mathcal{N}'$ is the set of points unique by their position, defined as:

$$\begin{cases} p_i \mid d(\mathbf{c}, p_i) \leq r \wedge (d_i, \theta_i, \phi_i) \notin \mathcal{U}, \mathcal{U} \leftarrow \mathcal{U} \cup \{(d_i, \theta_i, \phi_i)\} \\ \emptyset, \text{if } \forall (d_j, \theta_j, \phi_j) \in \mathcal{U}, |d_i - d_j| \geq \alpha \wedge |\theta_i - \theta_j| \geq \beta \wedge |\phi_i - \phi_j| \geq \gamma \end{cases} \tag{1}$$

$i = (i_1, \ldots, i_n), j = (j_1, \ldots, j_k), i, j, n, k \in \mathbb{Z}^+ n \leq |P|, k \leq |\mathcal{U}|$, d_i is Euclidean distance, θ_i is azimuthal angle [**ZHANG20211333**], ϕ_i is polar angle.

$$d(\mathbf{p},\mathbf{c}) = \sqrt{(p_x - c_x)^2 + (p_y - c_y)^2} \tag{2}$$

$$\theta(\mathbf{p},\mathbf{c}) = \arccos\left(\frac{p_y - c_y + \epsilon}{d(\mathbf{p},\mathbf{c}) + \epsilon}\right) \times \left(\frac{180}{\pi}\right),\ \text{if } 0 < \epsilon < 0.001 \tag{3}$$

$$\phi(\mathbf{p},\mathbf{c}) = \arctan 2\,(p_x - c_x, p_y - c_y) \times \left(\frac{180}{\pi}\right) \tag{4}$$

Let $P_{ri} = \mathit{proximity}(\mathbf{c}, \mathbf{P_I}, r)$, $P_{re} = \mathit{proximity}(\mathbf{c}, \mathbf{P_E}, r)$, $P_{re} \subset \mathbf{P_E}$, $P_{ri} \subset \mathbf{P_I}$

$$\text{Retrieve}(c, P_I, P_E, r) = \begin{cases} \text{Retrieve}(c, P_I, P_E, r + \Delta), \text{if } |P_{ri}| = |P_{re}| \\ (|P_{ri}|, |P_{re}|)\,, \text{otherwise} \end{cases}$$
$$\text{where } \Delta = \begin{cases} 0.369, & \text{if } |P_{ri}| + |P_{re}| > 3 \\ 0.123, & \text{otherwise} \end{cases} \tag{5}$$

Let $\mathbf{D}_i = \{d(\mathbf{c},\mathbf{p}) \mid \mathbf{p} \in \mathbf{P}_{ri}\}$ and $\mathbf{D}_e = \{d(\mathbf{c},\mathbf{p}) \mid \mathbf{p} \in \mathbf{P}_{re}\}$. The spatial probability [**SMITH2014**] for interior and exterior points relative to the centroid are calculated as:

$$\mathbf{S}(\mathbf{P_I} \mid \mathbf{c}) = exp\left(-\frac{1}{|\mathbf{D}_i|}\sum_{d \in \mathbf{D}_i} d\right) \tag{6}$$

$$\mathbf{S}(\mathbf{P_E} \mid \mathbf{c}) = exp\left(-\frac{1}{|\mathbf{D}_e|}\sum_{d \in \mathbf{D}_e} d\right) \tag{7}$$

$$\mathbf{S_i} = \mathbf{S}(\mathbf{P_I} \mid \mathbf{c}) = \frac{\mathbf{S}(\mathbf{P_I} \mid \mathbf{c})}{\mathbf{S}(\mathbf{P_I} \mid \mathbf{c}) + \mathbf{S}(\mathbf{P_E} \mid \mathbf{c})} \tag{8}$$

$$\mathbf{S_e} = \mathbf{S}(\mathbf{P_E} \mid \mathbf{c}) = \frac{\mathbf{S}(\mathbf{P_E} \mid \mathbf{c})}{\mathbf{S}(\mathbf{P_I} \mid \mathbf{c}) + \mathbf{S}(\mathbf{P_E} \mid \mathbf{c})} \tag{9}$$

The interior and exterior score of a point is calculated as:

$$\mathbf{I} = (\mathbf{S_i} \cdot \mathbf{w}) + \frac{|\mathbf{P}_{ri}|}{|\mathbf{P}_{ri}| + |\mathbf{P}_{re}|} \quad (10)$$

$$\mathbf{E} = (\mathbf{S_e} \cdot \mathbf{w}) + \frac{|\mathbf{P}_{ri}|}{|\mathbf{P}_{ri}| + |\mathbf{P}_{re}|} \quad (11)$$

$$\text{Refinement}\,(p,\ P_I, P_E, r, w) \text{ as } \mathbb{R}^2 \times \mathcal{P}(\mathbb{R}^2) \times \mathcal{P}(\mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$$

$$= \begin{cases} \text{Refinement}\,(p, P_I, P_E, 1.1r, w)\,, \text{if } F_i = F_e = 0 \\ [1, 0, 1]\,, \text{if } a + b = 1 \wedge a < b \wedge F_i = F_e \wedge \Psi \\ [1, S_i, S_e + 0.3]\,, \text{if } a + b \neq 1 \wedge a < b \wedge F_i = F_e \wedge \Psi \\ [1, 1, 0]\,, \text{if } a + b = 1 \wedge a \not< b \wedge F_i = F_e \wedge \Psi \\ [1, S_i + 0.3, S_e]\,, \text{if } a + b = 1 \wedge a \not< b \wedge F_i = F_e \wedge \Psi \\ [1, 0.5, 0.5]\,, \text{if } I + E < 3 \wedge (S_i = 1 \vee S_e = 1) \wedge p \in P_I \\ [2, 0.5, 0.5]\,, \text{if } I + E < 3 \wedge (S_i = 1 \vee S_e = 1) \wedge p \in P_E \\ [1, S_i, S_e]\,, \text{if } I + E < 3 \wedge ((S_i = S_e) \vee (S_i > S_e)) \wedge p \in P_I \\ [2, S_i, S_e]\,, \text{if } I + E < 3 \wedge ((S_i = S_e) \vee (S_i < S_e)) \wedge p \in P_E \end{cases} \quad (12)$$

$= (e1, e2, e3).e1 \in \{1, 2\}$ and $e2, e3 \in [0, 1]$ and $\Psi = ((F_i < 0.55 \wedge F_e < 0.55) \vee (|P_{ri}| + |P_{re}| < 3))$ and $(a, b) = \mathit{Retrieve}(\mathbf{c}, \mathbf{P_I}, \mathbf{P_E}, r)$

3 Results

The proposed model was evaluated using **sklearn.metrics** for binary classification. The results showed that the proposed model outperforms top existing models from SciKit library, and it's visual representation of the refined subsets shows a better class separation.

Method	TN	FN	TP	FP	ACC	PRE	REC	F1	MCC
PERSIST(proposed)	991.2	60.4	939.6	8.8	0.966	0.992	0.939	0.965	0.933
K-Nearest Neighbors	962.94	49.28	950.72	37.06	0.96	0.96	0.95	0.96	0.91
Random Forest	978.12	76.42	923.58	21.88	0.95	0.98	0.92	0.95	0.9
Decision Tree	967.82	90.88	909.12	32.18	0.94	0.97	0.91	0.94	0.88
Radius Neighbors	988.56	109.98	890.02	11.44	0.94	0.99	0.89	0.94	0.88
Kernel Density	931.72	169.62	830.38	68.28	0.88	0.92	0.83	0.87	0.77
Neural Net MLP	906.22	187.82	812.18	93.78	0.86	0.9	0.81	0.85	0.72

Table 1. Average performance of different models for data denoising

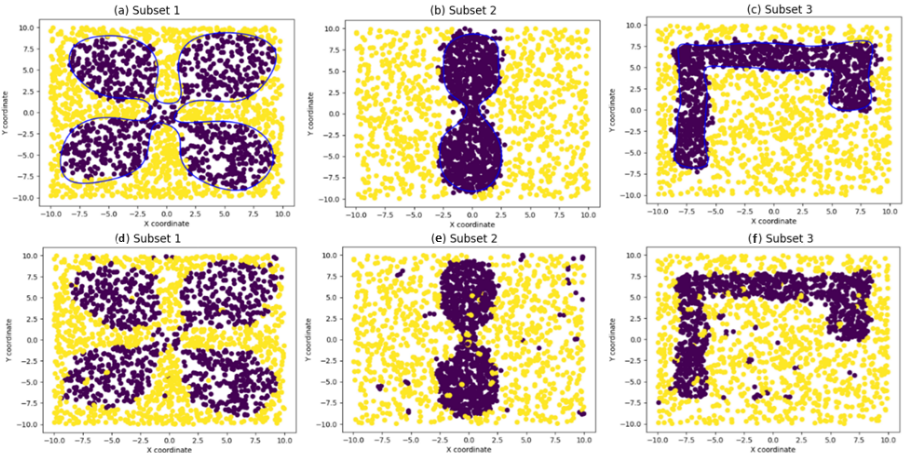

Figure 3. Persist model (a),(b),(c) and K-Nearest Neighbors (d),(e),(f)

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Mathematical Modelling of Shock Wave Reflection from a Heat-Conducting Wall

Grigore Secrieru

Abstract

In this work, on the basis of the model of the Navier-Stokes equations, a study of the dynamics of the flow, which arises in the process of reflection of a normally incident shock wave from a heat-conducting wall, is carried out. At the weak intensity of the shock wave and moderate jump in the initial temperatures of gas and wall, analytical expressions for the perturbation of gas parameters, describing the main characteristic features of the reflected flow, are obtained. This allows us to evaluate the influence of viscosity, thermal conductivity and other physical effects on the formation of dissipative and ideal non-viscous and non-heat-conducting zones in the region of the arising flow.

Keywords: the forming gas flow, model of the Navier-Stokes equations, small perturbations of parameters, viscous heat-conducting gas.

1 Introduction

The interaction of real gas with heat-conducting barriers, which are widely encountered in nature and everyday practice, leads to the development of non-stationary flows with a complex internal structure [1]. The process of flow formation is influenced by thermal conductivity and other physical effects [2], [3]. Taking into account the influence of physical effects is of interest for identifying the most important characteristics of the fields of arising flows in gas-solid contact systems. This makes it necessary to study flows, given the influence of physical effects accompanying the interaction of gas flows with various obstacles. The theoretical study of flow formation is based on generally accepted models of continuum mechanics. The system of Navier-Stokes equations is widely used for successfully modelling the laws of conservation of mass, momentum and energy [2].

The structure of the forming gas flow is studied on the example of solving the problem of reflection of a normally incident plane shock wave from a heat-conducting wall. At low intensities of the incident wave, gas-dynamic and thermal processes lead to the formation of a flow with small perturbations of parameters, which is described by the Navier-Stokes equations. It is supposed that the gas temperature behind the incident wave differs little from the initial wall temperature. The wall temperature distribution is modelled by the linear heat conduction equation [2]. Taking into account the smallness of the perturbation of the parameters of the arising flow, the study is carried out on the basis of the system of Navier-Stokes equations, linearized around the values of the parameters in the initial state. Analytical solutions to the problem in a linear formulation are obtained in order to analyze the formation of a continuous flow structure with the formation of dissipative and ideal non-viscous and non-heat-conducting zones.

2 Formulation of the problem and analysis of the solution

At the initial moment of time $t = 0$, a normally incident plane shock wave reaches the surface of an impenetrable, stationary heat-conducting wall, with an initial temperature T_w^0. At $t < 0$, the shock wave moved at a constant speed through a homogeneous resting gas, like in an ideal case. The falling wave is followed by a homogeneous flow of viscous heat-conducting gas, occupying a half-space $x > 0$ at $t = 0$, with constant parameters: speed u^0, pressure p^0, temperature T^0 and speed of sound c^0. Hereinafter, the upper index 0 is used to denote the values of the parameters at the initial moment of time, and the lower index w denotes the parameters of the wall.

Gas-dynamic and thermal processes of gas interaction with the wall at $t > 0$ lead to the emergence of non-stationary flows described by the system of complete Navier-Stokes equations, closed by the equation of state for a perfect gas $p = R\rho T$ (where R is the universal gas constant). The boundary conditions are set at a sufficient distance from the wall surface $x \longrightarrow \pm\infty$ and at the gas-wall contact surface ($x = 0$). As $x \longrightarrow \pm\infty$, the parameters are required to tend to the values given in the initial conditions. On the wall surface $x = 0$, conditions of adhesion and continuity of temperatures and heat flow are imposed.

The dynamics of the arising flow is studied in the case of a shock wave of weak intensity and moderate discontinuities in the initial temperatures of the gas and the wall $T_w^0/T^0 \sim 1$. This leads to the formation of flows with small perturbations of parameters. The solution to the problem is sought using its proximity to the solution at the initial moment of time. Each gas-dynamic parameter is represented as a sum of two values. The first value corresponds to the unperturbed value (given in the initial state). The second value expresses a small non-stationary perturbation, keeping the same notation for it (with index ϵ) as for the parameters themselves. After replacing the parameters by the sum of two quantities and discarding the terms with an order of smallness of perturbations higher than the first, we obtain a linearized system of Navier-Stokes equations for disturbances of gas parameters [2]:

$$\begin{gathered}
\frac{\partial \rho_\epsilon}{\partial t} + \rho^0 \frac{\partial u_\epsilon}{\partial t} = 0 \\
\rho^0 \frac{\partial u_\epsilon}{\partial t} + \frac{\partial p_\epsilon}{\partial x} = \mu^0 \frac{\partial^2 u_\epsilon}{\partial x^2} \\
\rho^0 c_p \frac{\partial T_\epsilon}{\partial t} = \frac{\partial p_\epsilon}{\partial t} + k^0 \frac{\partial^2 T_\epsilon}{\partial x^2} \\
p_\epsilon = R\rho^0 T_\epsilon + RT^0 \rho_\epsilon
\end{gathered} \quad . \tag{1}$$

Here μ is the viscosity coefficient, k is the thermal conductivity coefficient, c_p is the specific heat capacity at constant pressure.

Equations (1) and the equation for the wall temperature distribution form a system of partial differential equations with constant coefficients, which must be solved under appropriate initial and boundary conditions. An analytical solution can be obtained using the Laplace integral transform method with respect to the real variable t. In particular, the distribution of gas and wall temperatures is described by the expressions:

$$T(t,x) = T^0 + T_\epsilon(t,x), \tag{2}$$

$$T_w(t,x) = T_w^0 + T_{w\epsilon}(t,x). \tag{3}$$

Here, small temperature perturbations $T_{w\epsilon}(t,x)$ and $T_\epsilon(t,x)$ are determined from the analytical solution of the linearized problem. Expressions for other gas parameters can be found similarly [2].

The obtained analytical expressions (2), (3) for temperature and other gas parameters allow us to establish the main characteristic features of the emerging gas flow and the wall temperature distribution when a shock wave is reflected from a heat-conducting cold or hot wall. In this case, we can clearly trace the dynamics of the flow field structure development, as well as evaluate the influence of thermal conductivity and other dissipative factors. Based on the obtained analytical solutions, asymptotic expressions for the distribution of gas and wall temperatures for the most intense period of interaction at small times ($t << 1$) were found.

The obtained linearized expressions (2), (3) for the perturbation of temperature and other parameters can be considered as the first approximation of the perturbation of the parameters. Based on them, similarly, one can obtain the following approximation for the parameters of the forming one-dimensional gas flow.

3 Conclusion

On the example of solving the problem of reflection of a normally incident plane shock wave from a heat-conducting wall, an approximate method approach to the solution using the system of Navier-Stokes equations in the one-dimensional case is developed. For a weak intensity of the incident shock wave and a moderate jump in the initial temperatures of the gas and wall, analytical expressions are obtained for the perturbation of the gas parameters and wall temperature. The obtained linearized solution allows us to establish the main characteristic features of the formation of a continuous structure in the flow field of a viscous heat-conducting gas. In this case, it is possible to evaluate the influence of viscosity, thermal conductivity, and other physical factors on the formation of dissipative and ideal non-viscous and non-thermal conductive zones in the field of the forming gas flow. For the most intense period of interaction at small times, asymptotic expressions for the gas parameters and wall temperature are obtained.

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Using the Ellipsoid Method to Find Parameters of Lasso and Ridge Regressions

Petro Stetsyuk, Olha Khomiak

Abstract

We consider the optimization problem for finding the parameters of a linear regression according to the criterion of the least moduli powered to p $(1 \leq p \leq 2)$ with the regularization of parameters according to the criterion of the least moduli powered to q $(1 \leq q \leq 2)$. Its partial cases are lasso regression and ridge regression, as well as least squares method and the least moduli method. An algorithm for solving the problem is developed based on the well-known ellipsoid method.

Keywords: Lasso regression, ridge regression, linear regression, least moduli criterion, convex function, ellipsoid method.

1 Introduction

Lasso, ridge, and other linear regression models [1] belong to machine learning area and used in statistics, system identification, artificial intelligence problems, medicine, finance, economics, etc. These models are formulated as convex programming problems to which methods of minimizing non-smooth convex functions can be applied.

In the paper [2], the problem of minimizing a non-smooth convex function for finding the parameters of a linear regression model with the criterion of the least moduli powered to $1 \leq p \leq 2$ with L_1-regularization was formulated. Here we present a more general formulation of this problem, where the criterion of the least moduli powered to q $(1 \leq q \leq 2)$ will be used to regularize the parameters. The algorithm for solving the problem will use the ellipsoid method [3].

2 Formulation of the optimization problem

Let matrix $A = \{a_{ij}\}_{i=1,\dots,m}^{j=1,\dots,n}$, vector $y = \{y_i\}_{i=1,\dots,m}$, and scalars $\lambda \geq 0$, $1 \leq p, q \leq 2$ be given. Consider the convex optimization problem

$$f_{pq}^* = f(x_{pq}^*) = \min_{x \in \mathbb{R}^n} \left\{ f(x) = \sum_{i=1}^{m} \left| y_i - \sum_{j=1}^{n} a_{ij} x_j \right|^p + \lambda \sum_{i=1}^{n} |x_i|^q \right\}, \quad (1)$$

where $x = \{x_j\}_{j=1,\dots,n}$ is a vector of unknown parameters, and $|t|$ is the modulus (absolute value) of the number t. The function $f(x)$ is non-smooth if $p = 1$ or $q = 1$ and smooth if $p > 1$ and $q > 1$. Its subgradient at the point $\overline{x}$ is calculated by the formula

$$g_f(\overline{x}) = p \begin{pmatrix} \sum_{i=1}^{m} sign\left(\sum_{j=1}^{n} a_{ij}\overline{x}_j - y_i\right) \left|\sum_{j=1}^{n} a_{ij}\overline{x} - y_i\right|^{p-1} a_{i1} \\ \cdots \\ \sum_{i=1}^{m} sign\left(\sum_{j=1}^{n} a_{ij}\overline{x}_j - y_i\right) \left|\sum_{j=1}^{n} a_{ij}\overline{x} - y_i\right|^{p-1} a_{in} \end{pmatrix} +$$

$$+\lambda q \begin{pmatrix} \sum_{i=1}^{n} sign\left(\overline{x}_1\right) \left|\overline{x}_1\right|^{q-1} \\ \cdots \\ \sum_{i=1}^{n} sign\left(\overline{x}_n\right) \left|\overline{x}_n\right|^{q-1} \end{pmatrix}. \quad (2)$$

If $p = 2$, $q = 1$ or $q = 2$, then partial cases of the problem (1) is the well-known problem of minimizing a non-smooth function

$$f_{LASSO}^* = f_1(x_{2,1}^*) = \min_{x \in \mathbb{R}^n} \left\{ f_1(x) = \|y - Ax\|^2 + \lambda \sum_{i=1}^{n} |x_i| \right\}, \quad (3)$$

and the problem of minimizing a strictly convex smooth function

$$f_{RIDGE}^* = f_2(x_{2,2}^*) = \min_{x \in \mathbb{R}^n} \left\{ f_2(x) = \|y - Ax\|^2 + \lambda \sum_{i=1}^{n} x_i^2 \right\}. \quad (4)$$

Problem (3) uses L_1-regularization of unknown parameters and corresponds to lasso regression, and problem (4) uses L_2-regularization of unknown parameters and corresponds to ridge regression. If $\lambda = 0$, then the problem (1) corresponds to the least squares method ($p = 2$) and the least moduli method ($p = 1$).

3 Algorithm for solving the problem (1)

The problem (1) is a problem of unconditional minimization of a convex function $f(x)$. For its solving, Shor's ellipsoid method can be used, which is implemented as an **emshor** program [3]. We will show its application for the problem of the function $f(x)$ minimization, providing that its minimum point x^*_{pq} is localized in n-dimensional ball with radius r_0 that is centered at the point $x_0 \in \mathbb{R}^n$, i.e., $\left\|x_0 - x^*_{pq}\right\| \leq r_0$.

Algorithm 1 – Algorithm **emshor** for solving the problem (1)

Step 0. Choose $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ so that $\left\|x_0 - x^*_{pq}\right\| \leq r_0$. Moreover, choose $\varepsilon > 0$, set $B_0 := I_n$ and $k := 0$.

Step 1. Calculate $g_f(x_k)$ at the point x_k using formula (2).

Step 2. If $\left\|B_k^\top g_f(x_k)\right\| r_k \leq \varepsilon$, then set $k^* := k$, $x^*_\varepsilon := x_k$ and STOP.

Step 3. Calculate

$$x_{k+1} := x_k - h_k B_k \xi_k, \quad \text{where} \quad \xi_k := \frac{B_k^\top g_f(x_k)}{\left\|B_k^\top g_f(x_k)\right\|}, \quad h_k := \frac{1}{n+1} r_k.$$

Step 4. Update

$$B_{k+1} := B_k + \left(\sqrt{\frac{n-1}{n+1}} - 1\right) (B_k \xi_k)\, \xi_k^\top \quad \text{and} \quad r_{k+1} := \frac{n}{\sqrt{n^2-1}} r_k.$$

Step 5. Set $k := k+1$ and go to Step 1.

Theorem 1. *Let x_k and x_{k+1} be generated by Algorithm 1. Then, the ratio of volumes of the ellipsoids $\mathcal{E}_k = \{x \mid \|B_k^{-1}(x_k - x)\| \leq r_k\}$ and $\mathcal{E}_{k+1} = \{x \mid \|B_{k+1}^{-1}(x_{k+1} - x)\| \leq r_{k+1}\}$ does not depend on k and is equal to*

$$q_n := \frac{vol(\mathcal{E}_{k+1})}{vol(\mathcal{E}_k)} = \frac{n}{n+1} \left(\frac{n}{\sqrt{n^2-1}}\right)^{n-1} < \exp\left\{-\frac{1}{2(n+1)}\right\} < 1.$$

Moreover, $x^*_{pq} \in \mathcal{E}_k$ holds for all $k = 0, 1, \ldots, k^*$, $x^*_\varepsilon \in \mathcal{E}_{k^*}$ and, if Algorithm 1 stops, $f(x^*_\varepsilon) \leq f^*_{pq} + \varepsilon$ is valid.

4 Conclusion

In this paper, the convex optimization problem for finding the parameters of a linear regression according to the criterion of the least moduli powered to p $(1 \leq p \leq 2)$ with the regularization of parameters according to the criterion of the least moduli powered to q $(1 \leq q \leq 2)$ is formulated. An algorithm for solving the problem was developed based on the well-known ellipsoid method.

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Optimality Criterion and Algorithm for Solving the Traveling Salesman Problem

Dmitri Terzi

Abstract

Changing the direction of research from exact methods to methods such as genetics, with heuristic approaches, is promising for creating the desired algorithm. To this end, the paper presents a method for solving the traveling salesman problem, considering it as a transportation-type problem that can be easily solved by a modified distribution method. The main results are related to the expansion of the application of the latest technology. The criteria for optimality of the solution are presented. A new algorithm has been developed, as well as a new, fairly general, test problem generator with a given optimal solution, which is consistent with the hypothesis about the need to conduct mass computational experiments to build an effective way to solve the traveling salesman problem.

Keywords: signs of optimality of a solution, three-element replacement operation, modified distribution method, traveling salesman problem.

1 Introduction

The study relates to the traveling salesman problem, which consists of finding the optimal route that passes through each of the given points once and then returns to the starting point. The goal of the traversal is to minimize a function related to the route cost, distance traveled, or time spent [1]. According to its mathematical formulation, the traveling salesman problem is related to transport-type problems and can be used in other situations: to optimize the data transmission route between different nodes, in creating cost-effective production, drone mission planning, to develop more advanced software systems useful for technologies such as driverless cars, and in developing software for processing large amounts of information [2],[3],[4]. In continuation of the research [5], the goal here is to develop a new computational algorithm for solving the traveling salesman problem, based on the use of the three-element replacement operation and the technology of the modified distribution method.

2 Optimality criteria

Let us present two criteria for the optimality of the solution to the traveling salesman problem. One criterion is presented as a necessary feature of optimality, and the other as a sufficient condition for the optimality of the solution. Definition: The operation of three-element replacement is the replacement of three elements (i_1, j_1), (i_2, j_2), (i_3, j_3) of a given route with a cyclic sequence of other three elements. Let c be the matrix of transport costs, $x_{a_1b_1}, x_{a_2b_2}, ..., x_{a_nb_n}$ – arbitrary feasible route (solution) to the traveling salesman problem. We form combinations of three unit elements $x_{i_1j_1} = 1$, $x_{i_2j_2} = 1$, and $x_{i_3j_3} = 1$ in this solution and calculate

$$C(i_1, j_1, i_2, j_2, i_3, j_3) = c(i_1j_1 + c(i_2j_2) + c(i_3j_3).$$

Then for the optimal plan and any of its 3-element combinations, the following inequality must be satisfied

$$C(i_1, j_1, i_2, j_2, i_3, j_3 \leq \min(C(i_2, j_1, i_3, j_2, i_1, j_3, C(i_3, j_1, i_1, j_2, i_2, j_3).$$

If the inequality is not satisfied for a given solution, then it is not optimal, since after performing a three-element replacement operation $x_{i_2j_1} = 1$, $x_{i_3j_2} = 1$,$x_{i_1j_3} = 1$ or $x_{i_3j_1} = 1$, $x_{i_1j_2} = 1$,$x_{i_2j_3} = 1$ (depending on the order of the term on the right-hand side of the inequality at which the minimum is achieved), and putting $x_{i_1j_1} = 0$,

$x_{i_2j_2} = 0$,$x_{i_3j_3} = 0$, we will get the best solution. This three-element replacement operation is used to obtain a new feasible cyclic solution with a lower cost value, and the solution is again checked for finality. A necessary criterion for the optimality of a solution is that it is impossible to perform a three-element replacement operation for an optimal cyclic solution. Based on the optimality criterion, it is possible to construct schemes for solving the traveling salesman problem within the framework of the technology of the modified distribution method for solving the transport problem. At the same time, its calculation stages change significantly in connection with the requirement for the cyclicality of the route and the elimination of the degeneracy of the solution in the corresponding transport problem. Let us consider another criterion that is sufficient for the optimality of the cyclic set of $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ in the traveling salesman problem.

Theorem 1. *Let there be some vector $v = (v-1, v_2, ..., v_n)$ and the set of $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ satisfies the v-optimality condition: $c_{i_rj_r} - v_{j_r} \leq c_{i_rk} - v_k$ for all k. Then this set is a solution to the traveling salesman problem.*

A sufficient criterion can be used to test a given solution for optimality. To do this, a system of inequalities $c_{i_rj_r} - v_{j_r} \leq c_{i_rk} - v_k$ is constructed for each element of the $c_{i_rj_r}$. If the vector v can be determined from these inequalities, then the set $c_{i_1j_1}, c_{i_2j_2}, ..., c_{i_nj_n}$ is optimal, which follows from the cyclicity of the set and the equality

$$sum_{r=1}^{n} c_{i_rj_r} = sum_{r=1}^{n} v_{j_r} + sum_{r=1}^{n} (c_{i_rj_r} - v_{j_r}),$$

in which on the right-hand side one sum over v is constant, and the other is minimal due to the v-minimality of the elements $c_{i_rj_r}$, $r = \overline{1, n}$.

3 Solution of the traveling salesman problem as a special problem of the transport type

At the initial stage of Algorithm 1, two methods were used to obtain a non-degenerate cyclic solution: using the cost matrix ms (mc method) and the evaluation matrix mo (mo method). When using the matrix mc (mo), its elements were first ordered in ascending (descending) order, and then, based on the first element selected from the ordered sequence, the next element was selected, representing a cyclical continuation of the previous one. Such iterations are carried out until a final cyclic solution is obtained. At the stage of calculating potentials, the constructed solution must be supplemented with fictitious zeros to eliminate degeneracy and in such a way that it is possible to complete the process of determining all $2n$ potentials.

Algorithm 1.
1. Definition of the matrix $c_{i_r j_r}$ of order n.
2. Formation of the initial route (xv, yv), $f = sum_{i=1}^{n} c(xv_i, yv_i)$.
3. Start of iteration of searching for the optimal solution.
4. Generating a non-degenerate solution x corresponding to the transport problem and consisting of n unit elements $x(xv_i, yv_i) = 1$ and $n-1$ dummy zeros $x(xve_j, yve_j)$, $j = \overline{1,n}$. Calculation of potentials u_i and v_j, $i, j = \overline{1,n}$.
5. Definition of the evaluation matrix $muv = muv(u_i, v_j)$.
6. For each zero position (i, j) of the matrix x, $x_{ij} = 0$, and with a positive value of the evaluation matrix muv, $muv(u_i, v_j) > 0$, calculation of two cycles (1) and (2) is performed.
7. One cycle is built starting from cell zero to cell one in a row i_1,

$$(i_1^1, j_1^1), (i_2^1, j_2^1), ..., (i_k^1, j_k^1) \tag{1}$$

and another cycle starting from the zero cell in the direction of the fictitious zero in the row i_1,

$$(i_1^\epsilon, j_1^\epsilon), (i_2^\epsilon, j_1^\epsilon), ..., (i_k^\epsilon, j_k^\epsilon). \tag{2}$$

8. Recalculation of the initial solution (xv, yv) for each of the cycles.
9. Formation of new solutions (xm_1^1, ym_1^1) and $(xm_1^\epsilon, ym_1^\epsilon)$.
10. Checking the cyclicality of new solutions.
11. Selecting the best cyclical solution.

12. Finding the best solution based on (xm_1^1, ym_1^1) and $(xm_1^\epsilon, ym_1^\epsilon)$ solutions using 3-element replacement operations.
13. Based on the best cyclic solution found, we form a new initial solution (xv, yv).
14. Go to step 3, to the next iteration.
15. Steps 3–14 are repeated a specified number of times if we do not know in advance the optimal solution to the problem being solved; when solving test problems such that the optimal solution s known in advance, the iterations continue until the final solution is obtained.
The presented Algorithm 1 is a sequential application of 3-element substitution operations to solve the traveling salesman problem. The new cyclic solution is calculated by the cycle within the framework of the modified distributive method.

4 Construction of the traveling salesman problem with a predetermined optimal solution

Test problems are necessary to check the correctness of the calculation schemes and evaluate their efficiency. Algorithm 2 is based on the representation of the cost matrix as a decomposition into components. The cost matrix c is represented as the sum of three matrices: co is the optimality matrix, cb is the base matrix, and cr is the reduction matrix.

Algorithm 2.
1. Generate a cyclic substitution (xo, yo) that corresponds to the optimal route

$$(i_1, j_1), (i_2, j_2), ..., (i_n, j_n).$$

2. Formation of the optimality matrix co, in which the elements in positions $(xo(k), yo(k))$ are non-zero, as well as the diagonal elements,

$$co(i,j) = \begin{cases} -a, & \text{for } i = i_k, j = j_k, k = \overline{1,n} \\ Inf & \text{if } i = j \\ 0, & \text{in other cases} \end{cases}$$

where a is a positive integer constant.
3. Generate the base matrix cb with elements

$$cb(i,j) = \begin{cases} 0, & \text{for } i = i_k, j = j_k, \text{or } i = j, \\ b_{ij}, & \text{in other cases} \end{cases}$$

where b_{ij} are integer non-negative random numbers not less than a.
4. We define two random positive integer vectors p and q and a reduction matrix cr,

$$cr(i,j) = \begin{cases} p_i + q_j, & i \neq j, \\ 0, & i = j \end{cases}.$$

5. We construct the cost matrix c of the traveling salesman problem with a given optimal solution (xo, yo),

$$c = co + cb + cr.$$

Theorem 2. *Traveling salesman problem with cost matrix $c = co + cb + cr$, constructed according to Algorithm 2, has an optimal solution (xo, yo).*

Proof. From the construction of the matrix c it follows that, after reduction by rows and columns, in the matrix $co + cb = c - p - q$, in positions $(xo(i)), yo(i))$, $i = \overline{1,n}$, there are zeros, and in the remaining positions there are non-negative numbers. By construction, the sequence $(xo(i), yo(i))$ forms a cyclic route. Therefore, $(xo(i), yo(i))$, $i = \overline{1,n}$, is the optimal route of the traveling salesman problem with matrix c.

5 Conclusion

By successively applying three-element replacement operations in the current cycle of the modified distribution method, a good approximation to an acceptable solution occurs without large requirements for RAM. Computational experiments were carried out in various variants of generating the initial reference plan: using the above-described mc

method, the mo method, or with the selection of a random route, which was completed to the reference plan. If the new solution is better than the current one, it becomes the initial solution. This adaptation of the exact modified distributive method proves useful and is recommended for obtaining an acceptable solution to the traveling salesman problem of sufficiently large size in the allotted time.

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The Synthesis Function Method for Solving the Multi-Criteria Linear-Fractional Model in Integers

Alexandra Tkacenko

Abstract

In this paper, I will propose a method for solving the linear-fractional multi-criteria optimization model with identical denominators in whole numbers. The solving procedure involves assigning utilities (weights) to each criterion, after which a single-criterion linear-fractional optimization model is constructed and can be solved. By changing the utility values, we can determine a new optimal compromise solution(s). The algorithm is theoretically justified and it was tested successfully on several examples.

Keywords: Multi-criteria model in integers, efficient solution, optimal compromise solution.

1 Introduction

In many decision-making situations the optimal solution must be of the integer type. A number of studies can be listed here, such as: dynamic memory allocation, multiprocessor systems, general positioning problems. But when the problem is of multi-criteria non-liniar type, this effort increases considerably, but even more so when the solution must be of integer type. Currently there is an increased interest in such types of models, especially from a practical point of view.

2 Problem formulation

The integer multi-criteria linear-fractional optimization problem is described as follows:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \min \\ \max \\ x \in D \end{array} \right\} F_k\left(x\right) = \frac{\sum_{j=1}^{n} c_{kj} x_j}{\sum_{j=1}^{n} d_j x_j}, \quad k = \overline{1, r} \\ A \cdot x \leq b \\ x \in Z^+ \end{array} \right. , \tag{1}$$

in which: $D = \left\{ x = \left(x_1, x_2, \ldots, x_n\right)^T \left| Ax \leq b, x \in Z^+ \right. \right\}$, $A = \|a_{ij}\|$ is an array of size $m \times n (m < n)$, $C = \|c_{kj}\|$, is an array of size $r \times n(r < n)$, d is a n-dimensional line vector. x is an n-dimensional column vector and b is an m-dimensional column vector.

In order to solve the model (1), we impose the next condition:

$$\sum_{j=1}^{n} d_j x_j \neq 0, \quad (\forall)\, x \in D.$$

3 Findings and theoretical justifications

The multi-criteria optimization models, as a rule, do not admit an optimal solution. The solution(s) is one that achieves the best compromise of all criteria, also called Pareto optimal, non-dominated, efficient [2]. We will present some ways of defining them for model (1).

1. The solution $x^* \in Z^+$ for the model (1) is the vector that optimizes a synthesis function of all r criteria, i.e.: $h(F) = h[F_1, F_2, ..., F_r]$.

2. The solution $x^* \in Z^+$ is the vector which minimizes a single criterion such as: $\phi(x^*) = \min_{x \in D} h(\Psi_1(x - X_1), ..., \Psi_1(x - X_r))$, in which $X_j = (x_{1j}, x_{2j}, ..., x_{nj})^T, j = \overline{1, r}$ is the optimal solution of each criterion, F_j, and Ψ_k is a distance type function between the vector $x \in D$ and optimal solution X_k for each criterion F_k.

3. The solution $x^* \in Z^+$ is a vector which belongs to a set of efficient solutions of integer type.

Definition 1. *The basic solution* $\overline{X} \in Z^+$ *of the model (1) is a basic efficient one if and only if for any other basic solution* $X \in Z^+$*, where* $X \neq \overline{X}$*, for which the relations* $F_{j_1}(X) \geq F_{j_1}(\overline{X})$ *are true, where* $j_1 \in (1, ..., j_2)$ *– indices corresponding to the maximum type of criteria, there immediately exists at least one index* $\exists j_l \in (j_2 + 1, ..., r)$*, of the minimum type for which is true the relation:* $F_{j_l}(X) > F_{j_1}(\overline{X})$ *or, if the relation* $F_{j_l}(X) \leq F_{j_1}(\overline{X})$ *is true for all indices corresponding to the minimum type of criteria which are* $j_l \in (j_2 + 1, ..., r)$*, immediately exists at least one index from the set of indices of the maximum type of criteria* $\exists j_1 \in (1, ..., j_2)$*, for which the next relation* $F_{j_2}(\overline{X}) < F_{j_2}(X)$ *is true.*

4 The Combinatorial synthesis algorithm

One of the most important problems that arises when solving the multi-criteria optimization problem in integers is: what kind of optimal solutions of each criterion we will use to build the synthesis function of all criteria, these being in R^+ or in Z^+? In order to answer to this question, we propose to apply the next combinatorial method.

Stage I.

1. At this stage, it is necessary to solve $2r$ unicriteria linear fractional programming problem from model (1), of which r are of the type: $F_j = optim_{x \in D} F_j(x)$ and the other r – of the type: $F_j^p = pessim_{x \in D} F_j(x)$ on the same admissible domain:

$$D = \{x \in R | Ax \leq b, x \geq 0\};$$

2. Next, we will analogically solve $2r$ linear fractional programming problems of integer type [1].

3. We will build the vectors of records of the optimal and of the worst values of the objective functions, using in each combinatorial vector both values of some criteria in Z^+ and of others in R^+.

Stage II.

1. Randomly considering one of the vector records of the optimal values of the objective functions and correspondingly of the worst values, we construct the synthesis function of the model [2], which ex-

presses the summary utility of all criteria thus: $G = \sum_{j=1}^{r} (\alpha_j F_j + \beta_j)$, We will maximize this function on the same admissible domain, using the utility maximization method [2]. The obtained optimal solution is one of the best compromise for model (1).

Theorem 1. *For any utility values assigned to the objective functions in model (1), where the identical denominator is nonzero over the admissible domain, the optimal compromise solution(s) corresponding to them remains the same for any combinatorial selection of the optimal and its corresponding pessimistic values of the criteria in R^+ or in Z^+.*

5 Conclusion

The proposed method is valuable in order to determine the optimal compromise solution(s) in intege of the model (1), because it allows combinatorial selection of the values of each objective function in R^+ or in Z^+. Last but not least, the decision maker has the free choce of celecting utility values criteria. This fact makes it possible to solve model interactively, increasing both efficiency and its attractiveness.

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